Shape Function Analysis of Three Dimensional Pre-Tensioned Spherical Dome

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Abstract

Predicting the shape function of a pre-tensioned spherical dome is very important for the safety of a dome structures design and performance under cyclic loading. This paper presents a new shape function mathematical model which is proposed for use with three dimensional pre-tensioned spherical dome that should provide for a more accurate stresses. The new model is proposed based on an improved Airy stress function principle; to retain objectivity of the results for three dimensional pre-tensioned spherical dome. This model is adopted in this study for its simplicity and computational efficiency. The objective of this work is to analyze the response and to describe the behavior of pre-tensioning dome structure under loading. The model provides a very powerful tool for the solution of many problems in elasticity; such applications include tensor analysis of the stresses and strains. Correlation between the proposed model with experimental studies results of pre-tensioned specimens are conducted and show a reasonable agreement. The results are drawn as to the applicability of this approach. Stresses within dome surface are constant and the shear stress is zero when subjected to a hoop stress. The maximum stress occurs at the boundary of the dome intersecting the y-axis and is decreased along the boundary of the disc as it nears the x-axis. The maximum compressive stress occurs at the boundary intersecting with the x-axis and decreases as it nears the z-axis along the interfacing boundary.

Keywords : Airy stress function ,Pre-stressing , Pre-tensioning, Finite-element method, Nonlinear analysis, Slip

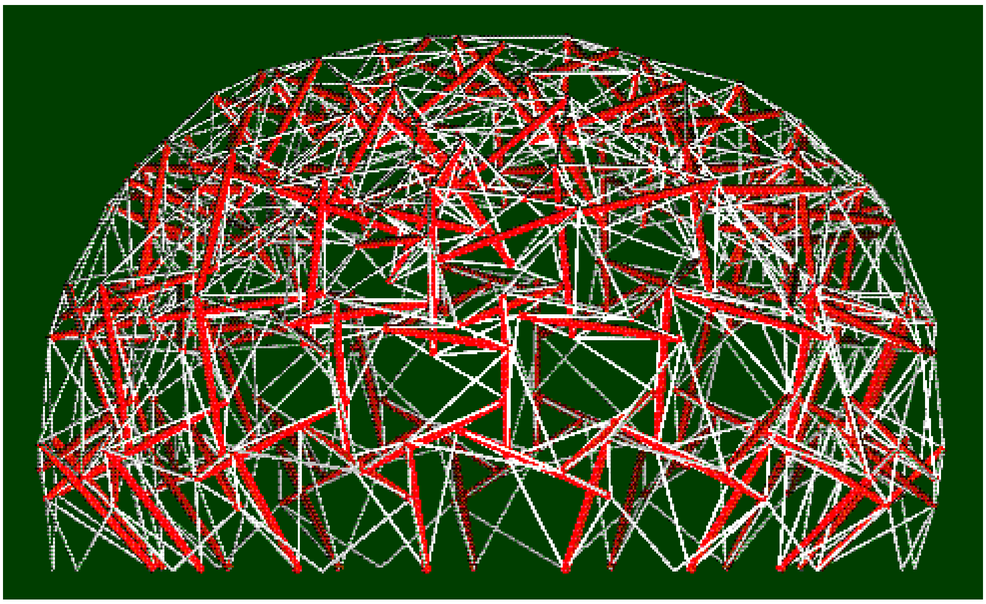
المستخلص

ان التنبا لدالة شكل القبة المستديرة التي لها قاعدة دائرية الكروية ثلاثية الابعاد المشدودة مسبقا مهم جدا في التصميم والاداء الامن لهيكل القبة المستديرة تحت ظروف اداء احمال دورية مختلفة. في هذا البحث تم اقتراح نموذج رياضي لدالة شكل جديدة يمكن استخدامها في القبة المستديرة التي لها قاعدة دائرية الكروية ثلاثية الابعاد المشدودة مسبقا والتي يمكن الاستفادة منها لايجاد الاجهادات بدقة اكبر. ان النموذج المقترح يستند على دالة ايري المعدلة للاجهادات للمحافظة على

فعالية ايجاد النتائج للقبة المستديرة التي لها قاعدة دائرية كروية ثلاثي الابعاد مشدود مسبقا وهذا النموذج تم اعتماده في هذه الدراسة لبساطته وكفائتة في ايجاد الحسابات حيث ان الهدف من هذا العمل هو تحليل الاستجابة ووصف سلوك القبة الكروية تحت تاثير الاحمال وهو اداة قوية لحل المسائل في المرونة .تم استخدام اسلوب الموتور في تحليل الاجهادات والانفعالات وان الربط بين نتائج الدراسات العملية وان نتائج النموذج المقترح بينت التوافق المعقول بينهما من خلال رسم النتائج لبيان قابلية الطريقة. الاجهادات داخل سطح القبة الكروية ثابت وإجهاد القص صفرا عندما يتعرض لإجهاد حلقي. ان الضغط الأقصى يحدث في حدود القبة الكروية متقاطعا على المحور الصادي وينخفض على طول الحدود من القرص لأنها تقترب من محور س وان اجهاد الضغط الاقصى يحدث في أقصى الحدود التي تتقاطع مع المحور س وتقل لأنها تقترب من محور ص على طول رابط الحدود.

1. Introduction

Roth and Whitely [1] propose a technology based on tensegrity for tough, rigid, large scale domes that are also economical to construct. The development of a structural technology to economically cover large areas would be useful for warehouses, permanent or temporary protection for archaeological and other vulnerable sites, large-scale electrical or electromagnetic shielding and exclusion or containment of flying animals or other objects. Structures based on such a technology can serve as frameworks in which environmental control, energy transformation and food production facilities could be embedded. The space application is also possible by using self-deployed structures. Summary advantages are improved rigidity, ethereal, resilient, equal-length struts, simple joints.



Figure(1). A representation of a dome which utilizes tensegrity solutions' technology [2].

Predicting the shape function of a pre-tensioned spherical dome is very important for the safety of a dome structures design. In addition, there are growing concerns about the performance dome under loading. Hence, a more realistic evaluation of the dome structure behavior due to cyclic loading is necessary to maintain containment integrity.

Elasticity is an elegant and fascinating subject that deals with the determination of the stress, strain and distribution in an elastic solid under the influence of external forces. A particular form of elasticity which applies to a large range of engineering materials, at least over part of their load range produces deformations which are proportional to the loads producing them, giving rise to the Hooke's Law. The theory establishes mathematical models of a deformation problem, and this requires mathematical knowledge to understand the formulation and solution procedures. The variable theory provides a very powerful tool for the solution of many problems in elasticity. Employing complex variable methods enables many problems to be solved that would be intractable by other schemes. The method is based on the reduction of the elasticity boundary value problem to a formulation in the complex domain. This formulation then allows many powerful mathematical techniques available from the complex variable theory to be applied to the elasticity problem [3].

Material properties must be determined experimentally. Careful examinations of the properties of most structural materials indicate that they are not isotropic or homogeneous. Nonetheless, it is common practice to use the isotropic approximation for most analyses. In the future of structural engineering, however, the use of composite, anisotropic materials will increase significantly. The responsibility of the engineer is to evaluate the errors associated with these approximations by conducting several analyses using different material properties [4].

In the recent years, thin shell structures find wide applications in many branches of technology such as space vehicle, nuclear reactor, pressure vessels, roofs of industrial building and auditoriums. From the point of view of architecture, the development of shell structure offers unexpected possibilities and opportunities for the combined realization of functional, economic and aesthetic aspects studied and tested conical concrete-shell specimens with widely varying material properties and traced their load deformation response, internal stresses and crack propagation through the elastic, inelastic, and ultimate stress ranges [5].

The finding of a structural evaluation of the 5 meter diameter observatory dome structure constructed by Observa-Dome laboratories, Inc. Regarding to the presentation of literature review, it should be emphasized that no investigation related to the analysis of large concrete thin shell dome is found. So it can be represented this work as a first one in the field of the study of concrete dome. The main objectives of this study are conducting an analytical study on the behavior of reinforced concrete ribbed dome with precast rib and cast-inplace cover and concrete slab under monotonically increasing loads by using three dimensional finite element method of analysis [6].

2. The airy stress function

A stress function is a function from which the stress can be derived at any given point x, y. These stresses then automatically satisfy the equilibrium conditions.
Now let’s examine such a stress function. The Airy stress function ϕ is defined by [7]

$$\sigma_x = \frac{\partial^2\phi}{\partial y^2}, \quad \sigma_y = \frac{\partial^2\phi}{\partial x^2} \quad \text{and} \quad \tau_{xy} = -\frac{\partial^2\phi}{\partial x\,\partial y} \tag{1}$$

One can insert these stresses in the equilibrium conditions (1.1). One then directly see that they are satisfied for every how convenient... However, if one inserts the above definitions into the compatibility condition, we get

$$\frac{\partial^4\phi}{\partial x^4} + 2\frac{\partial^4\phi}{\partial x^2\,\partial y^2} + \frac{\partial^4\phi}{\partial y^4} = 0 \quad \text{or} \quad \nabla^4\phi = 0 \tag{2}$$

This equation is called the biharmonic equation. It needs to be satisfied by every valid Airy stress function as well.

3. Pre-tensioned structures

Pre-tensioned high strength trusses using alloy steel bar are widely used as glass wall supporting systems because of the high degree of transparency. The breakage of glass panes in this type of system occurs occasionally, likely to be due to error in design and analysis in addition to other factors like glass impurity and stress concentration around opening in a spider system. Most design does not consider the flexibility of supports from finite stiffness of supporting steel or reinforced concrete beams [8]. The resistance of lateral wind pressure of the system makes use of high tension force coupled with the large deflection effect, both of which are affected by many parameters not generally considered in conventional structures. In the design, one must therefore give a careful consideration on various effects, such as support settlement due to live loads and material creep, temperature change, pre-tension force, and wind pressure. It is not uncommon to see many similar glass wall systems fail in the wind load test chambers under a design wind speed. This paper presents a rigorous analysis and design of this type of structural systems used in a project in Hong Kong, China. The stability function with initial curvature is used in place of the cubic function, which is only accurate for linear analysis. The considerations and analysis techniques are believed to be of value to engineers involved in the design of the structural systems behaving nonlinearly [9].

4. Shape function of three dimensional pre-tensioned spherical dome: A model

In this paper, a prediction of the stresses of the dome structure model is made through various types of numerical modeling, taking in account the appropriate non-linearity for each material. For the nonlinear finite element analysis, the dome is idealized as an axisymmetric model and a three dimensional global model. In order to simulate the actual behavior of the dome, both numerical models are refined by comparison of the results of the two analyses and with the existing research results. Furthermore, more recently developed material models for dome are introduced to the model.

The shape function for three-dimensional pre-tensioned spherical dome logically provides a reasonable stresses estimate. These shape functions are based on an n-order in general approximation, which provides for a non-linear interpolation. For three-dimensional simulation this function offers advantages over the linear shape function currently used in two-dimensional simulations [9]

$$\begin{aligned}\phi(x,y,z) = {} & A_o + A_1xy + A_2y^2 + A_3x^3 + A_4x^2y + A^5xy^2 + A_6y^3 + A_7x^4 + A_8x^3y \\ & + A_9x^2y^2 + A_{10}\,xy^3 + A_{11}\,y^4 + A_{12}\,z + A_{13}xyz + A_{14}\,xz^2 + \cdots \\ & + A_i\,x^n y^n z^n \end{aligned} \tag{3}$$

A_o, A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8, A_9,… A_i Constants determined from the boundary conditions

Which is satisfy Airy Stress Function (Compatibility condition) equations (1, 2)

5. Tensor analysis

Tensors are geometric objects that describe linear relations between vectors, scalars, and other tensors. Elementary examples include the dot product, the cross product, and linear maps. Vectors and scalars themselves are also tensors. A tensor can be represented as a multi-dimensional array of numerical values. The order (also degree or rank) of a tensor is the dimensionality of the array needed to represent it, or equivalently, the number of indices needed to label a component of that array. For example, a linear map can be represented by a matrix, a 2-dimensional array, and therefore is a 2nd-order tensor. A vector can be represented as a 1-dimensional array and is a 1st-order tensor. Scalars are single numbers and are zeroth-order tensors.

∇= Vector differential operator (Del or Nabla)

Gradient φ = Grad φ =∇ φ

Divergence φ =Div φ=∇. φ

Curl φ =Rotation φ =Rot φ =∇ Xφ

Compatibility of strain tensor

$$\nabla\nabla.\varepsilon_{ij} = \nabla^{t}\nabla.\varepsilon_{ij} - \nabla\nabla e - \Delta\varepsilon_{ij} = 0 \qquad (4)$$

and

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}SI;\ \mathrm{I} = \mathrm{unity}\ ; = \frac{1-2\nu}{E}S \qquad (5)$$

Where e=trace of strain, S=trace of stress, ∇^{t}=gradient transpose

Trace of stress

$$S = \sigma_x + \sigma_y + \sigma_z$$

Trace of strain

$$e = \epsilon_x + \epsilon_y + \epsilon_z$$

σ_{ij}= Stress tensor

ε_{ij} =Strain tensor

$$\nabla.\varepsilon_{ij} = \frac{1+\nu}{E}\nabla.\sigma_{ij} - \frac{\nu}{E}\nabla S \qquad (6)$$

$$\square e = \frac{1-2\nu}{E}\square S \qquad (7)$$

$$\Delta\varepsilon_{ij} = \frac{1+\nu}{E}\Delta\sigma_{ij} \qquad (8)$$

$$\square^{T}\square S = \square\square S \qquad (9)$$

$$\square\square.\varepsilon_{ij} = \frac{1+\nu}{E}\square\square.S - \frac{\nu}{E}\square\square S \qquad (10)$$

$$\square\square e = \frac{1-2\nu}{E}\square\square S \qquad (11)$$

$$\square\square.S + \square^{T}\square.S - \Delta\varepsilon_{ij} - \frac{1}{1+\nu}\square\square S + \frac{\nu}{1+\nu}\Delta S.I = 0 \qquad (12)$$

$$\sigma_{ik,jk} + \sigma_{jk,ik} - \Delta\sigma_{ij} - \frac{1}{1+\nu}\square\square S + \frac{\nu}{1+\nu}\Delta S.I = 0 \qquad (13)$$

From Beltrami

$$\Delta S + \frac{1}{1-\nu}\square.f_i = 0 \qquad (14)$$

f_i=body forces

External forces

$$T_{ij} = \sigma_{ij}n_j$$

Tensorial relation

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\delta_{ij}S \qquad (15)$$

$$\sigma_{ij} = \lambda\delta_{ij}e + 2G\varepsilon_{ij} \qquad (16)$$

δ_{ij}=Kroniker delta (unit tensor)

I=unity ; $I = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$

$\delta_{ij} = \begin{cases} 0 & \text{for} \quad i \neq j \\ 1 & \text{otherwise} \end{cases}$

Modulus of rigidity

$$G = \frac{E}{2(1+\nu)}$$

E=Young's modulus

ν=Passion's ratio

Lame's coefficients

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \quad (17)$$

$$\gamma_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \quad (18)$$

γ_{ij} =Strain tensor

$u_{i,j}$ = Displacements gradient for Cartesian coordinate

$$\gamma_{xx} = \epsilon_x = \frac{\partial u}{\partial x}, \gamma_{yy} = \epsilon_y = \frac{\partial v}{\partial y}, \gamma_{zz} = \epsilon_z = \frac{\partial \omega}{\partial z} \quad (19)$$

$$\gamma_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \quad (20)$$

$$\gamma_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial \omega}{\partial y}\right) \quad (21)$$

$$\gamma_{zx} = \frac{1}{2}\left(\frac{\partial \omega}{\partial x} + \frac{\partial u}{\partial z}\right) \quad (22)$$

x,y,z=1,2,3

u,v,ω=1,2,3

$\boldsymbol{\Omega_{ij}}$ **= Rotational strain tensor**

$$\Omega_{ij} = \frac{1}{2}(u_{i,j} - u_{j,i}) \tag{23}$$

$$\Omega_x = \frac{1}{2}(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial z}) \tag{24}$$

$$\Omega_y = \frac{1}{2}(\frac{\partial u}{\partial z} - \frac{\partial \omega}{\partial x}) \tag{25}$$

$$\Omega_z = \frac{1}{2}(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}) \tag{26}$$

6. Result and discussion

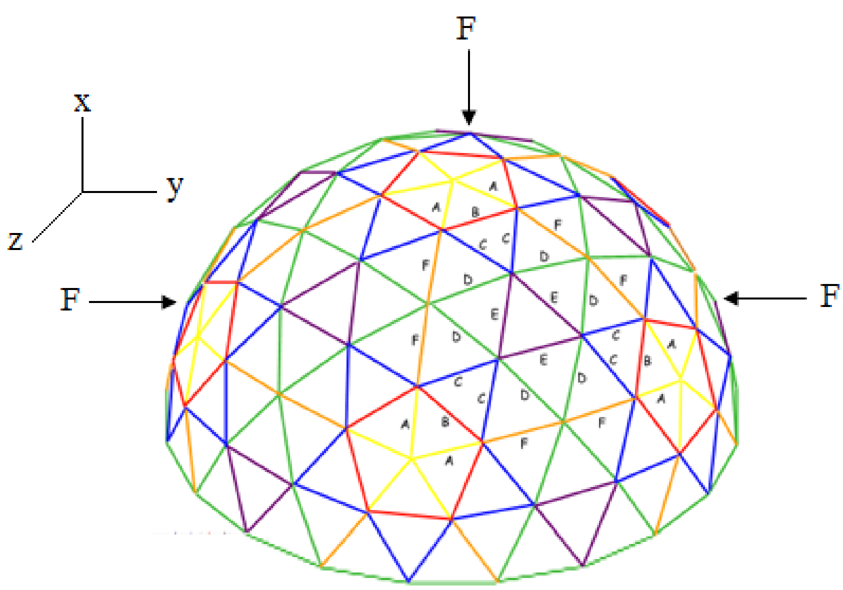
The stresses of a spherical dome have then been analyzed using shape function. The outputs are calculated and plotted against the added loading. Dynamic analysis of three-dimensional dome structural is a direct extension of static analysis. The elastic stiffness matrices are the same for both dynamic and static analysis. It is only necessary to lump the mass of the structure at the joints. The addition of inertia forces and energy dissipation forces will satisfy dynamic equilibrium. The dynamic solution for steady state harmonic loading, without damping, involves the same numerical effort as a static solution.

The performance of this new formulation has been tested through a variety of linear and nonlinear mechanical problems. In all of these tests, the new shape function showed better performance than the previous formulation. In particular, the improvement is significant in the three-dimensional dome structural test. Figure(2-4) shows the geometry, loading ,boundary conditions ,orientation and coordinates system for this problem.

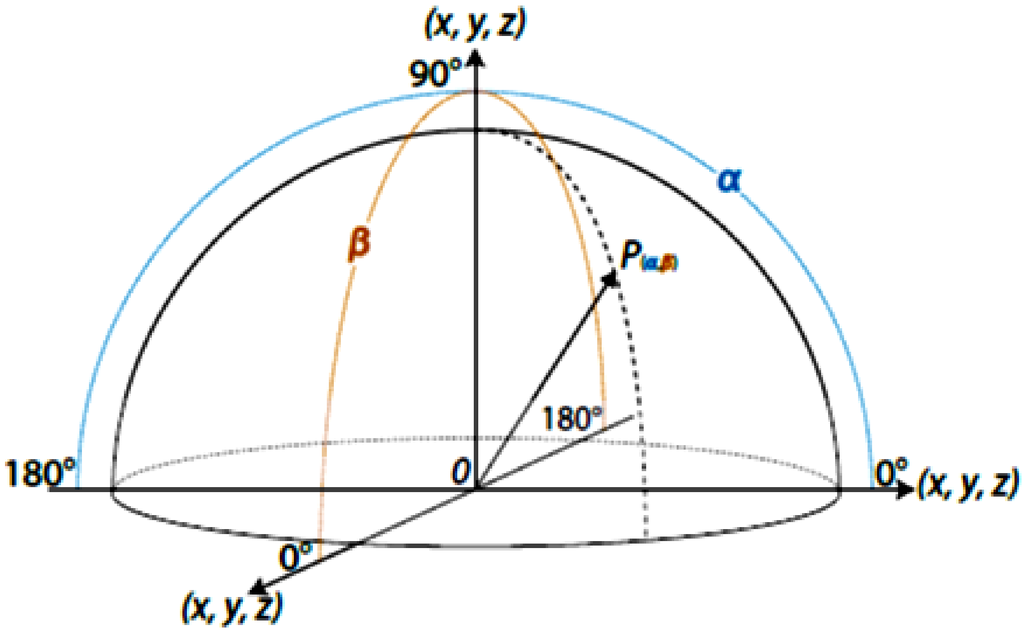
The radius is R=20m, the thickness t=0.7m, Young's modulus $E=8.3\times10^5$ and Poisson's ratio **ν**= 0.4. Due to symmetry the three dimensional pre-tensioned spherical dome is meshed using a single element through the thickness and with three unit loads along directions Ox, Oy and Oz The element is the standard, multi-node, full integration solid element. The results are reported in Table 1 in terms of the normalized displacement at the load point and then the output used to simulate the model in the Nastran Program.

Table (1). Shape Functions for a Nine-Node 20 Element .

Optional Nodes	Shape Function $\phi(x, y, z)$	Normalized Displacement		
		u	v	w
3	xy	1	1	-1
4	xy^3	-1	1	0
5	xy^2	0	-1	-1
6	y^4	1	0	1
7	xy^3	0	1	0
8	xyz	-1	0	-1
9	xz^2	0	0	0



Figure(2). Dome .



Figure(3). Spherical dome orientation.

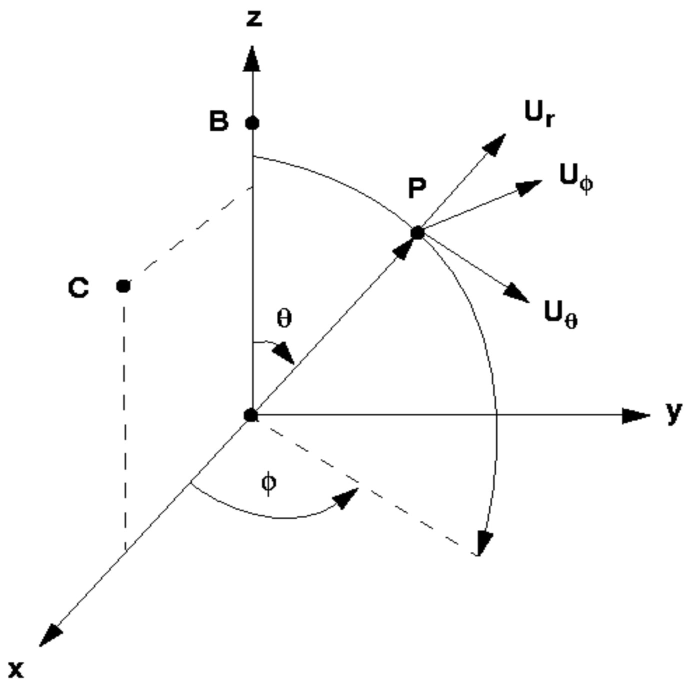
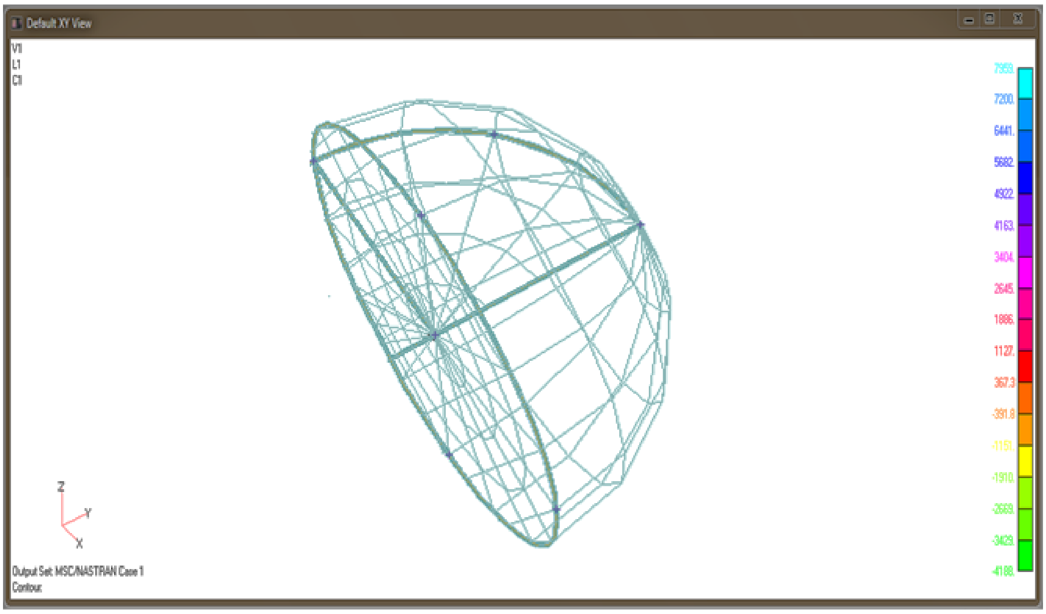
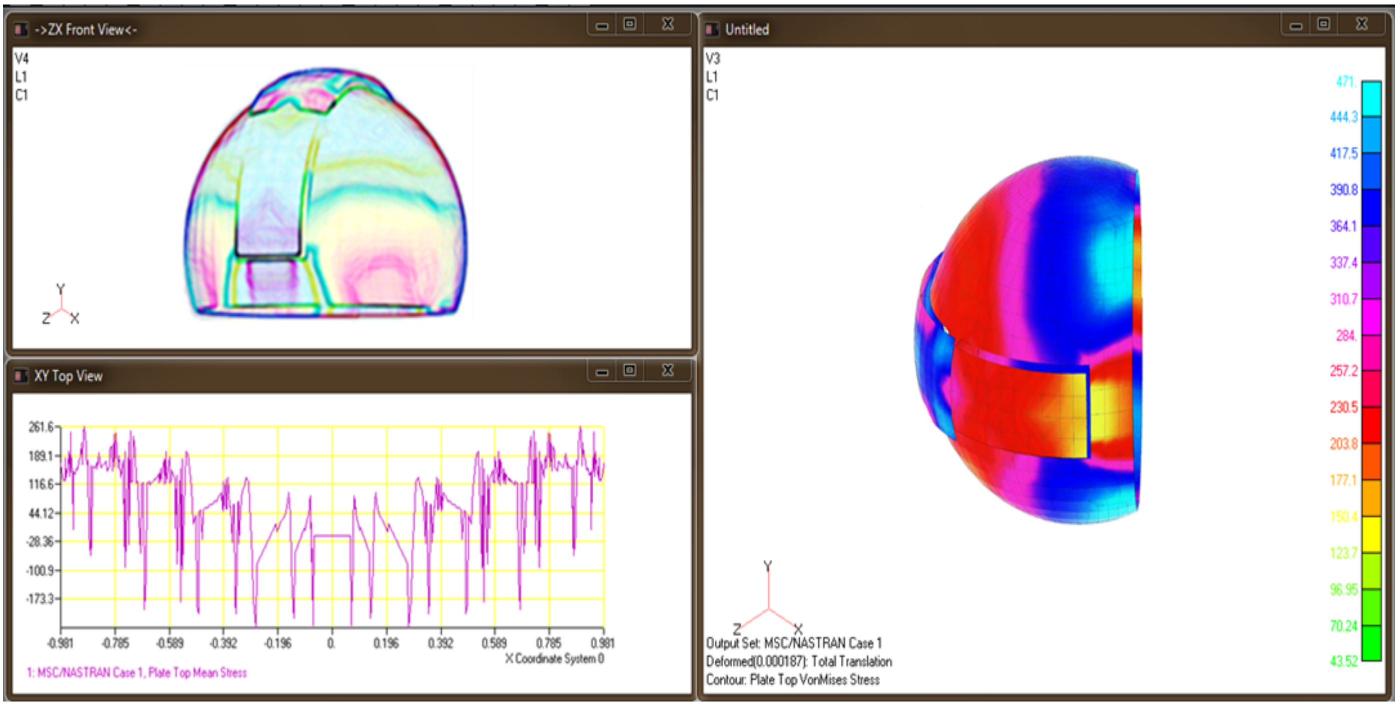


Figure (4). Spherical Coordinate Systems.

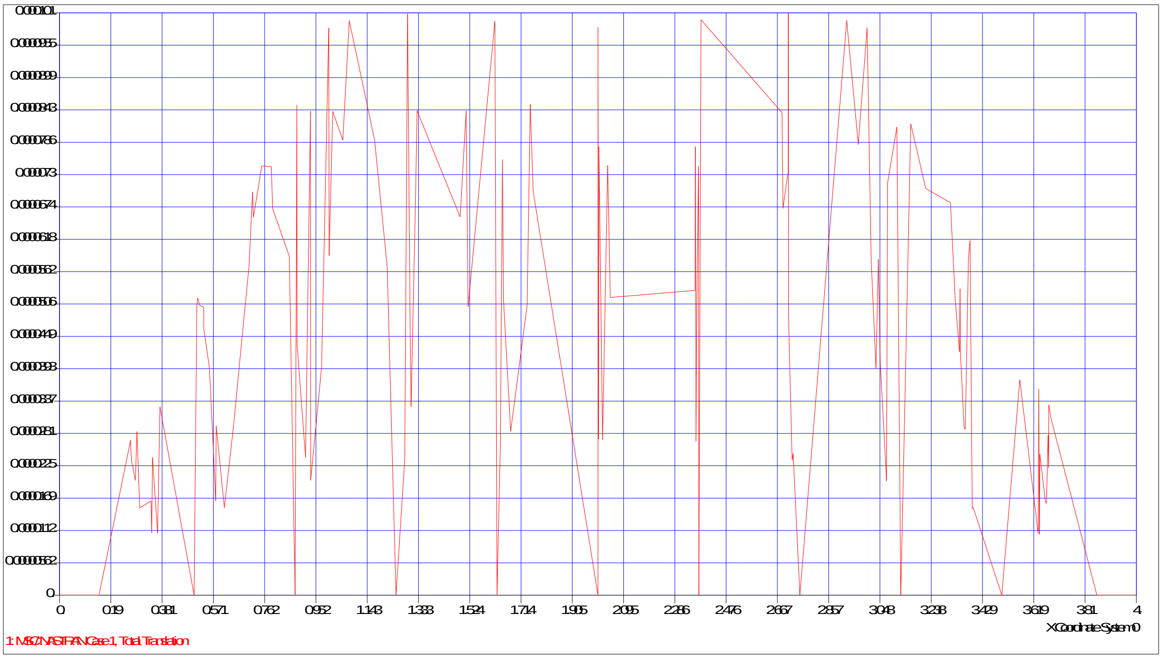
The solution of shape function is obtained with multi element; the nonlinear response to the initial radius is accurately recovered with (nxn) mesh of elements for a hemispherical dome. The model output used in the Nastran Program to build the dome structure, to simulate the dome stresses and deformation, and to draw stresses within total translation, within total rotation and maximum shear stresses, which is indicated in Figure (5-10)



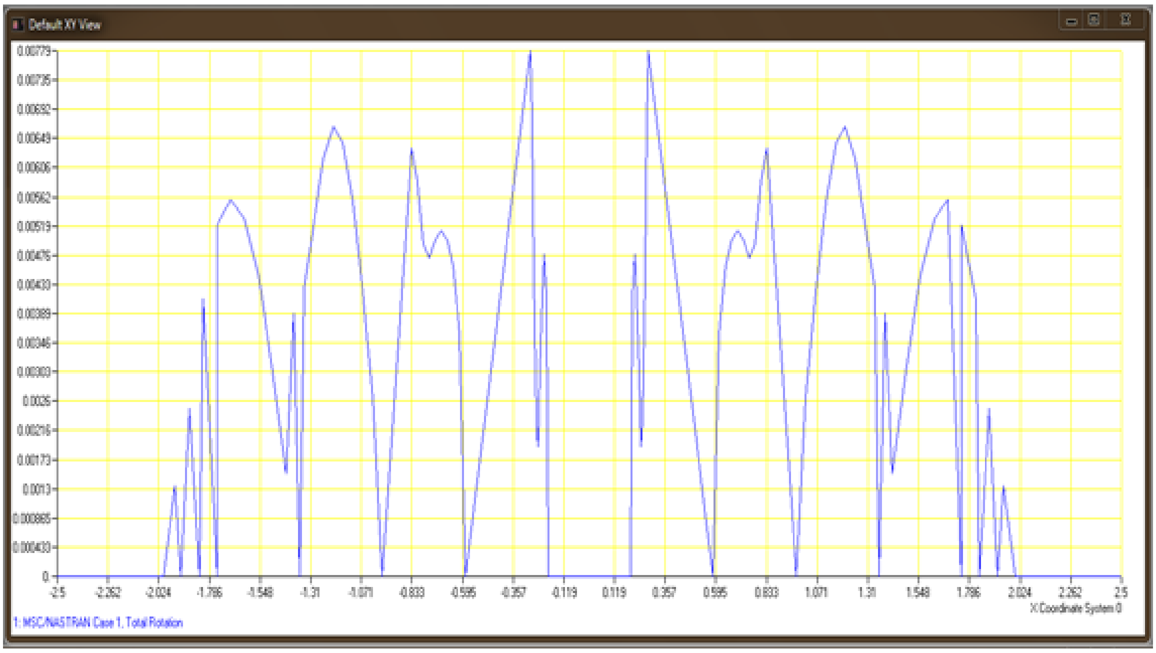
Figure(5). Output MSC NASTRAN Dome Building.



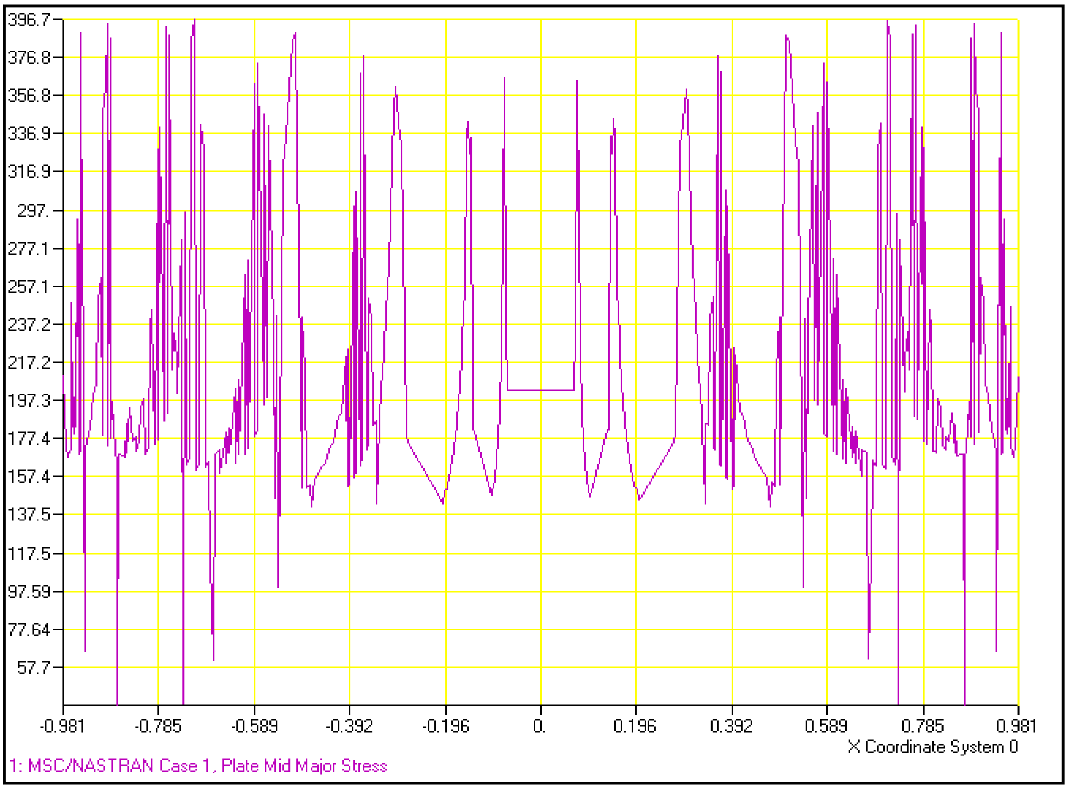
Figure(6). Output MSC NASTRAN Dome stresses and deformation.



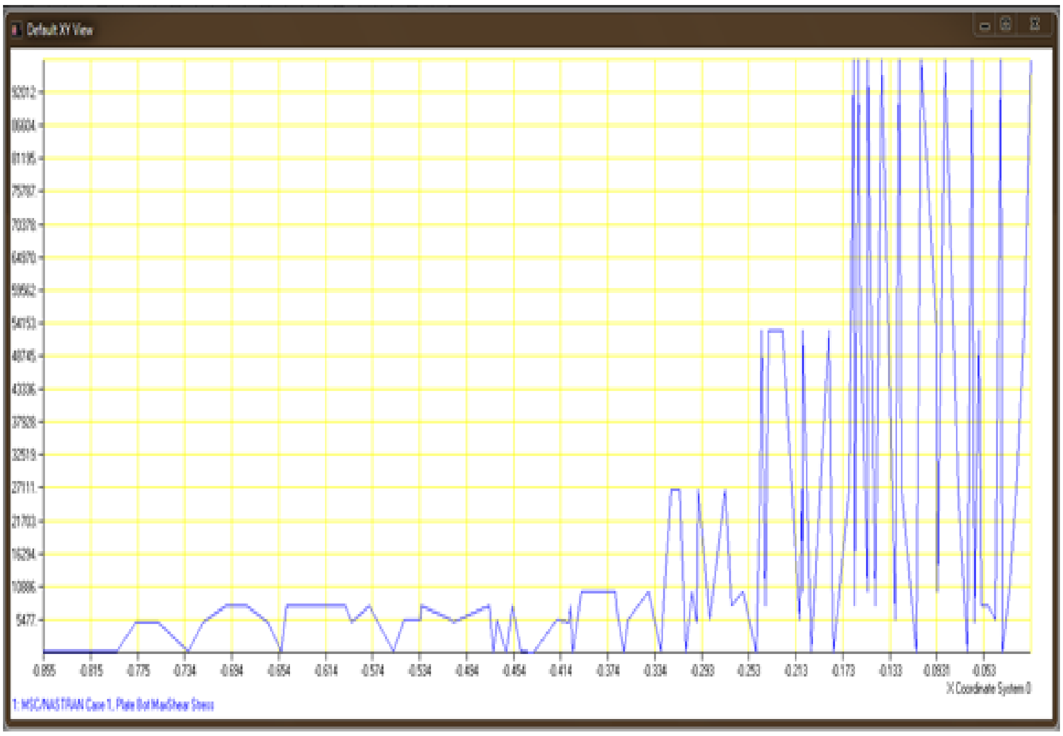
Figure(7). Stresses within total translation.



Figure(8). Stresses within total rotation.



Figure(9). Stresses in the x-direction.



Figure(10). Max shear stresses.

7. Conclusion

This study demonstrated how to solve an elasticity problem using the proposed shape stress function. It showed how the method can be applied to find the stresses and displacements at any point on a 3-dimensional dome subjected to different boundary conditions. This led to how this stress function can be applied to any phase dome in finding the stresses and displacements at any point.

The aims of this paper is to enhance understanding of shape function of dome structures from theoretical point of view and to provide insights into the problems associated with computational modeling of their structural form and behaviour. The most commonly used computational approach is described, together With a brief evaluation of the method

1-Output stress resultants, shear forces and moments for dome structure elements is a required analysis output for any plate and shell type structure.

2- Displacements could be specified in a spherical coordinate system. This simplified the enforcement of boundary conditions on axisymmetric models.

3- Demonstrated how the stress function is applied to the spherical dome. On studying the graphical representation of the result, it can be seen that all stresses within dome surface are constant and the shear stress is zero when subjected to a hoop stress. The maximum stress occurs at the boundary of the dome intersecting the y-axis and is decreased along the boundary of the disc as it nears the x-axis. The maximum compressive stress occurs at the boundary intersecting with the x-axis and decreases as it nears the z-axis along the interfacing boundary.

4- The stresses in the spherical dome are three dimensional at the top of the spherical dome; the value of the circumferential strain is equal to that of the radial strain because the stretching is uniform in all directions at the apex. Furthermore, the circumferential strain is fixed at zero at the edge of the spherical dome, due to the clamping condition at the boundary.

8. References

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9. Nomenclature

$\phi(x, y, z)$	Shape function, [dimensionless]
σ_x	Stress in x-direction, [N/m^2]
σ_y	Stress in y-direction, [N/m^2]
τ_{xy}	Shear stress in xy direction, [N/m^2]
T_{ij}	External forces, [N]
σ_{ij}	Stress tensor, [N/m^2]
ε_{ij}	Strain tensor, [N/m^2]
δ_{ij}	Kroniker delta (unit tensor), [dimensionless]
E	Young's modulus, [N/m^2]
ν	Passion's ratio, [dimensionless]
λ	Lame's coefficients, [dimensionless]
γ_{ij}	Shear Strain tensor, [dimensionless]
$u_{i,j}$	Displacements gradient for Cartesian coordinate, [mm]
Ω_{ij}	Rotational strain tensor, [dimensionless]
∇	Vector differential operator (Del or Nabla), [dimensionless]