Studying the Effect of Scale of Fluctuation on the Flow Through an Earth Dam Using Stochastic Finite Element

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Abstract

Estimation of quantity of flow through earth dams by using finite elements method mainly depends on the magnitudes of permeability for the earth dam. The permeability of soil medium may be included in the solution as constant permeability or non homogeneous, anisotropic or may be distributed as random field. Simulation of properties of soil by using random fields' generators has great interest nowadays. This paper includes studying of the effect of anisotropy of scale of fluctuation of the permeability through the stochastic finite elements analysis for case of flow through earth dam. Permeability has been simulated by random field and mapped into finite element methods according to the most recent studies. The results have shown that (1) quantity of seepage through stochastic earth dam is always less than that calculated for deterministic state. (2) As horizontal to vertical scale of fluctuation (anisotropy of scale of fluctuation) increases, quantity of seepage increases but it still less than that for deterministic state. The seepage elevation is also reduced at random field state compared to deterministic solution.

Keywords: Flow; stochastic finite element, earth dam; scale of fluctuation.

دِراسَة تأثير مِقياسِ التقلّبِ على التدفق خلال سدٍّ ارضي باستعمال عناصرِ محدودة عشوائية

المستخلص

تقدير كميةِ التدفق خلال **السدودِ الأرضِية** بإستعمال طريقةِ العناصرِ المحدودةِ تَعتمدُ بشكل رئيسي على مقاديرِ النفاذيةِ **للسد الترابي**. نفاذية وسطِ التربةِ قَدْ تُتضمّنُ في الحَلّ كنفاذية ثابتة أوغير متجانسة غير **متماثلة أَو قَدْ تُوزّعُ كحقل عشوائي**. **محاكاة خصائص التربةِ بإستعمال مولّداتِ الحقولِ العشوائيةِ** لَها إهتمامُ عظيمُ هذه الأيامُ. يتضمن هذا البحث دِراسَة تأثيرِ **اللاتماثل** لمِقياسِ تقلّبِ النفاذيةِ خلال تحليلِ عناصرِ محدودة عشوائية **لحالةِ التدفق خلال سد ارضي**. النفاذية تم محاكاتها **بحقل عشوائي وخطّطتْ إلى طرقِ العنصرِ المحدودةِ طبقاً لآخر الدِراساتِ. النَتائِج** بينت بأنّ (1) كمية **التسرب خلال سد ترابي عشوائي** دائماً أقل مِنْ تلك المَحْسُوبةِ للحالةِ الحتميةِ. (2) زيادة مقياس التقلب الافقي الى العمودي (مِقياس **اللاتماثل**

للتقلّب) يزيد من كمية التسرب لكنّها ما زالتْ أقل مِنْ التي للحالةِ الحتميةِ. إنّ إرتفاعَ التسرب ينخفضُ أيضاً عند الوسط العشوائي مُقَارَنة مع الحالةِ الحتميةِ.

1. Introduction

It is well known that properties of soil at the site are different from point to point. Permeability is the important soil property that used in calculation of quantity of seepage in different projects such as seepage through earth dam. Studying the variability of permeability of soil through the soils has revealed that permeability may be simulated as a random field. Design of earth dam most widely considers the average values (first moment of statistics) of permeability in case of different values are found for small regions in the earth dam medium. If the permeability simulated as random field, standard deviation and correlation are necessary to be used.

Finite element method is combined with random filed concept in two ways: directly included in finite element formulation such as those researched by (Freeze, 1975, Dagan 1993,) [1,2] or indirectly included to finite element code in which the random field is manipulated in isolated form then mapped to the finite element such as those developed by (Fenton and Griffiths, 1993) [3]. A good review on stochastic finite element can be found on **Stefanou** (2009) [4].

Spatial variability of soil property such as permeability has been investigated by Griffiths and Fenton 1993, Popescu, et, al., 2005, Hoeksema 1985, Ahmed 2009 [5,6,7,8]. These studies have shown that flow rates calculated based on permeability simulated as random field are less than flow rates calculated based on average permeability solution which is called deterministic solution. This conclusion has a great attention in design and it increases the importance for further studies on the effect of scale of fluctuation "correlation through distance". Very few researchers study the effect of anisotropy of scale of fluctuation on the rate of flow. For instance Ahmed, 2009 [8] reported that anisotropy of permeability decreases the flow rate for a simple dam with core. In spite of that the subject still need more researches and more range of scale of fluctuation should be included in calculations.

This research has used fixed mean and variable standard deviation for soil permeability property to study the effect of permeability on the rate of flow through earth dam utilizing the finite element method plus random field generation for permeability property. They are merged together to get a stochastic finite element analyses (Griffiths and Fenton, 1993) [5]. The main objective of this study is to investigate the effect of random field permeability on the flow rate. Variance and spatial correlation effects on the quantity of seepage are investigated also. Anisotropy of scales of fluctuation effects are also investigated since it mostly represents the real case of soil in the field.

2. Mean and covariance

It is well known from site investigations that soil properties differ from point to point. If these properties are simulated as random field, they can be characterized by first and second moment of statistic. In spit of that classic design uses just the first moment "mean value" which can be expressed mathematically as

$$u = \frac{\sum x_i}{N} \tag{1}$$

Where u is the mean value; N is the number of events; x is the event.

However the second moment of statistic (Equation 2) has an effect and should be taken specially when a lot of data concerning a property are

$$\sigma_x = \sqrt{\frac{x_i - u}{N}} \tag{2}$$

Sometimes the coefficient of variation is used instead of variance which is equal to standard deviation to the mean value. It is reasonable to say that soil properties through short distance are more correlated than long distance. Spatial correlation is measured by scale of fluctuation which is defined as the distance over which the properties show appreciable correlation (Jaksa et. al., 1999) [9].

Stochastic model

Very few researches have been done in the investigation of spatial variation of soil properties in a stochastic finite element method. It has been applied for different applications in geotechnical engineering such as flow through earth dam. It consists of two parts: the first is an ordinary two dimensional finite element model for flow through earth dam developed by Smith and Griffiths (1998)[10] and the second is the random field model for permeability property. The results of random field permeability are mapped to the finite element model (Fenton and Griffiths,1996) [3].

Flow through earth dam is governed by Laplace's equation.

$$k_x \frac{\partial^2 \phi}{\partial x^2} + k_y \frac{\partial^2 \phi}{\partial y^2} = 0 \tag{3}$$

where ϕ is fluid potential and k_x , k_y are permeability in x, y direction.

The Laplace differential equation has been implemented in a program that has been developed by several researchers such as Smith and Griffiths (1998) [10]. The program can compute the free surface location and flow through the earth dam using a two-dimensional iterative finite element model for constant or variable permeability. Finite element program developed to solve problems without random field permeability such as flow through earth dam was developed originally by Smith and Griffiths, (1998) [10]. Another development in analysis of anisotropy of permeability of typical earth dam or core installed in large earth dam was by Shakir (2004), Shakir, (2009) and Shakir, (2011)[11,12,13]. The combination of finite element with random field has been developed by Fenton and Griffiths, 1996 [3].

3. Permeability field generation

Permeability is considered to be log normally distributed in accordance with the field measurements as determined by Freeze (1975), Hoeksema and Kitanidis (1985), Sudicky, (1986) [1,7,14] and reported by Griffiths and Fenton (1993) [5]. Permeability is expressed as K(x) where x is any point in the two dimensional field of soil. Variance and "σ_{lnk}^2" mean "u_{lnk}^2" are the first and second moment of statistics for the normally distributed "ln k". These measures are represented by the following two equations.

$$\sigma_{\ln k}^2 = \ln\left(1 + \frac{\sigma_k^2}{\mu_k^2}\right) \tag{4a}$$

$$\mu_{\ln k} = \ln(\mu_k) - \frac{1}{2}\sigma_{\ln k}^2 \tag{4b}$$

It is reasonable to think that the permeability value at a point is highly correlated with the permeability at close point and in contrast the permeability at a point is different from permeability at another point far from the first point "poorly correlated". The distance between the two points is called scale of fluctuation. Correlation structure can be modeled depending on field observations. Generally, It can be defined mathematically as following (Sudicky, 1986) [14].

$$\rho(\tau) = \exp\left\{-\frac{2}{\theta}\tau\right\} \tag{5}$$

Where ρ is the correlation coefficient between two points separated by a large distance τ. Above function is a function of the distance between two points ($\tau = x_i - x_j$). This model indicates that correlation ρ decay exponentially. θ is the scale of fluctuation which is defined as the distance beyond which the field is effectively uncorrelated (Vanmarke 1984) [15]. This relation is modeled depending on expensive observation of permeability value in the field. Confirming this equation is difficult since it depends on large data which are expensive to be obtained. Figure (1) demonstrates the effect of increasing the distance τ with respect to θ on the correlation structure ρ. For instance, when θ is equal to 50 the correlation coefficient, ρ, is equal to 0.135 at distance τ=50.

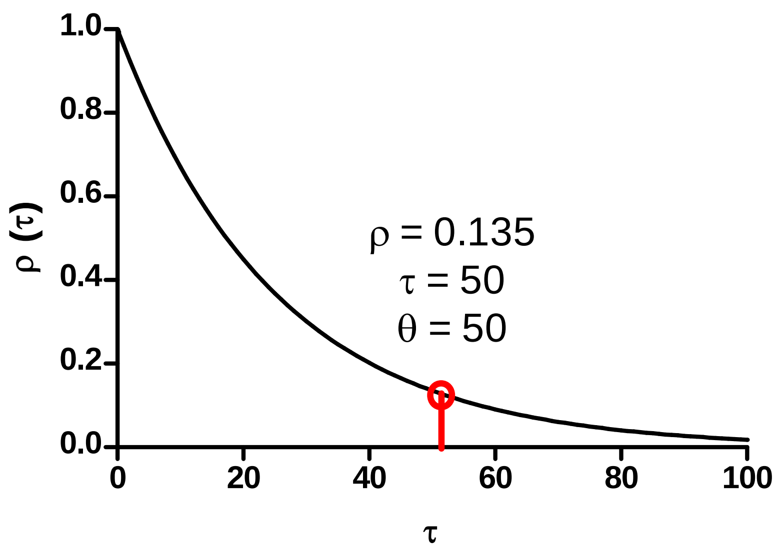

Figure (1). Effect of distance on the correlation of permeability property (Eq. 5).

If the permeability properties are observed at field, the problem is simple. We just mapped these values in the finite element program. Mostly large number of permeability values is not available but we can get mean and variance and also the correlation structure. According to that we can get the distribution by Monte Carlo simulation to give a picture of permeability through the field.

$$k_i = \exp\{\mu_{\ln k} + \sigma_{\ln k} g_i\} \tag{6}$$

Where u_{lnk} is the mean and σ_{lnk} is the standard deviation of the lognormal distributed k_i.

Uniform permeability on each realization can be obtained as scale of fluctuation goes to infinity. In contrast the permeability is different rapidly from element to element as scale of fluctuation goes to zero. G_i is derived from the standard Gaussien random field g through using LAS (Local average subdivision developed by Fenton and Vanmark 1990 [16] technique with zero mean and unit variance and a spatial correlation controlled by the scale of fluctuation (Vanmarcke 1984) [15].

Application

The size of the dam is shown in Figure (2). The base width was 20 m, the top width was 2 m and the height was 10 meter. Through random field generation the permeability has different values from element to element. About 605 program executions were performed. Table 1 shows a list of sets of values of scale of fluctuation and covariance used in this study. On the other hand the value of quantity of seepage is calculated for the same problem using mean value of permeability. For case of k =0.1 cm/s based on the solution of Gilbowy, (Harr, 1962) [17] Δ=9 m; 0.3Δ = 2.7 m; d= 13.7 m; α = 48^o, m =0.42; q = kmh = 0.1 (0.42)(10) = 0.42; q/(kh) =0.42. The finite element based solution used in this study gives approximately the same result (q/kh=0.42) .

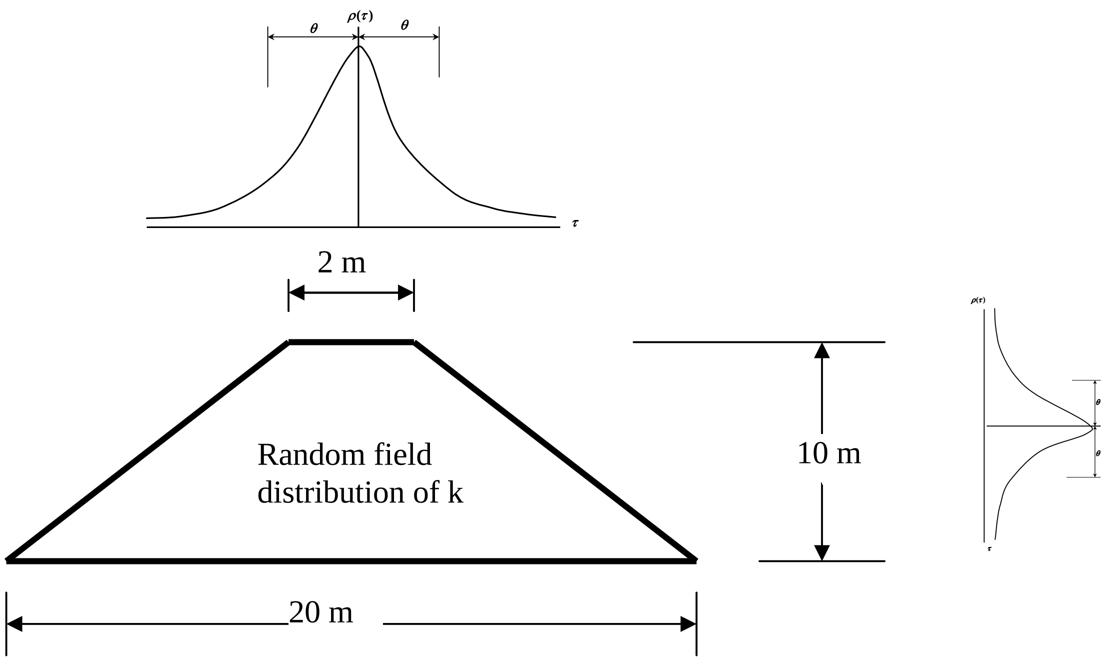

Figure (2). Dimensions of earth dam.

Table (1). Set values included to the stochastic program.

σ_k/u_k	θ_x	θ_y	σ_k/u_k	θ_x	θ_y	σ_k/u_k	θ_x	θ_y	σ_k/u_k	θ_x	θ_y	σ_k/u_k	θ_x	θ_y
0.1	0.1	0.1	0.1	0.1	0.8	0.1	0.1	3.2	0.1	0.1	6.4	0.1	0.1	12
0.2	0.2	0.1	0.2	0.2	0.8	0.2	0.2	3.2	0.2	0.2	6.4	0.2	0.2	12
0.4	0.4	0.1	0.4	0.4	0.8	0.4	0.4	3.2	0.4	0.4	6.4	0.4	0.4	12
0.8	0.8	0.1	0.8	0.8	0.8	0.8	0.8	3.2	0.8	0.8	6.4	0.8	0.8	12
1.6	1.6	0.1	1.6	1.6	0.8	1.6	1.6	3.2	1.6	1.6	6.4	1.6	1.6	12
3.2	3.2	0.1	3.2	3.2	0.8	3.2	3.2	3.2	3.2	3.2	6.4	3.2	3.2	12
6.4	6.4	0.1	6.4	6.4	0.8	6.4	6.4	3.2	6.4	6.4	6.4	6.4	6.4	12
12	12	0.1	12	12	0.8	12	12	3.2	12	12	6.4	12	12	12
24	24	0.1	24	24	0.8	24	24	3.2	24	24	6.4	24	24	12
48	48	0.1	48	48	0.8	48	48	3.2	48	48	6.4	48	48	12

4. Effect of covariance and scale of fluctuation

The problem of flow through the earth dam shown in Figure (2) has been solved for two cases. The first case includes using the specified permeability in all elements of the mesh and the second includes using permeability as random field. In literature the first one may be called deterministic solution and the second is undeterministic solution.

The normalized quantity of flow (q/kh) that obtained from deterministic solution (4) contains using average permeability. Figure (3) shows the relationship between normalized quantity of seepage versus covariance (σ_k /μ_k). As covariance amount increases, normalized quantity of seepage decreases. This conclusion is compatible with solution of Ahmad, (2009) [8] and also the solution of Griffiths and Fenton (1993) [5]. This means that allowable quantity of seepage is in the safe side and is greater than semi real case of random field. The flow in any studied cases will not increase more than the deterministic state.

In the same previous figure, using fixed scale of fluctuation in y direction θ_y = 0.1 and using different scales of fluctuation in x direction θ_y = 0.1 to 96 causes different results. Increasing of the scale of fluctuation in x direction increases the normalized quantity of seepage especially when covariance, (σ_k/u_k) exceeds the value of 0.7. Decreasing of scale of fluctuation causes high probability to existence of elements with small permeability and increasing the scale of fluctuation causes small probability to existence of element with low permeability. This may interpret the difference between the two cases of using scale of fluctuation. Increasing of Cov, (σ_k/u_k) more than 0.1 makes the normalized quantity of seepage decreases. This is reasonable since the high permeability variance affect flow more than low variance. High variance in permeability gives high variance in seepage.

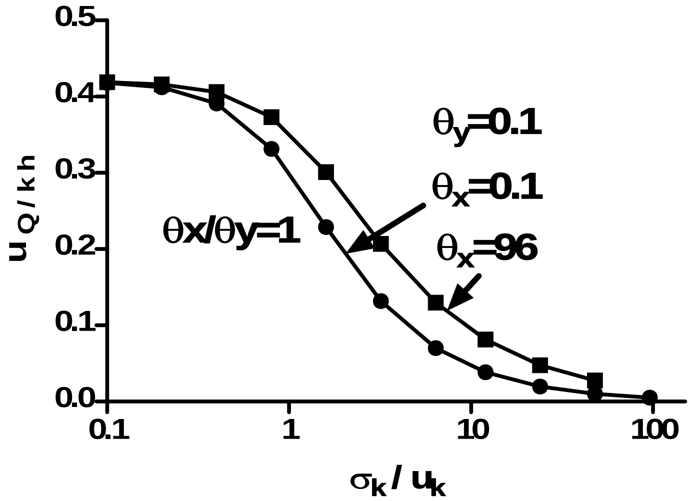

Figure (3). Relation between normalized quantity of seepage and covariance.

Figure (4) shows the relationship of the standard deviation of the quantity of seepage versus the covariance (standard deviation to mean value) of permeability. The standard deviation increases as correlation factor increases at covariance equal 1.6 the curve decays i.e. the standard deviation decreases. For instance at $\sigma_k / \mu_k = 0.4$ the standard deviation σ_Q equals 0.0275 at $\theta = 0.1$ while it reaches 0.048 (maximum) at $\sigma_k / \mu_k = 1.6$ after that it decreases. Scale of fluctuation has an effect on the σ_Q as it increases the standard deviation increases. This is may be attributed to increase the potential of existence elements with high difference in quantity of seepage value. This phenomenon is reported previously by Griffiths and Fenton, (1993) [5] and the result in this section confirms this phenomenon for wide range of sale of fluctuation.

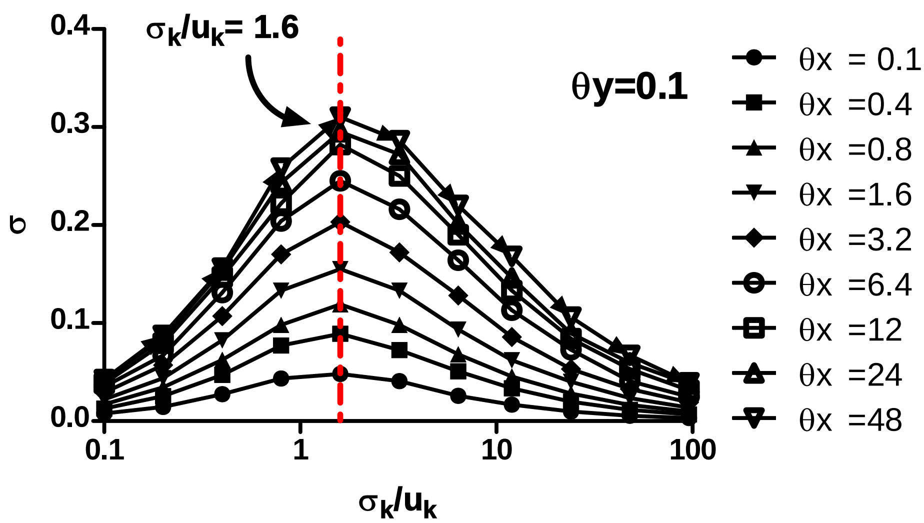

Figure (4). Quantity of seepage variance versus covariance.

5. Scale of fluctuation

Figure (5) shows the relationship between normalized quantity of seepage versus scale of fluctuation in x direction, $\theta_{x,}$ for a selected value of scale of fluctuation (the spatial correlation length) in y direction (0.1; 0.8, 3.2 and 6.4. The covariance, COV, (σ_k/u_k) was selected as 3.2. As the scale of fluctuation increases, the quantity of seepage increases. The scale of fluctuation is interpreted as the distance over which correlation is significant. Since many of the statistics of interest depend strongly on the scale of fluctuation, different scales of fluctuation are implemented. Some studies have considered the permeability and scale of fluctuation for reconstituted soil used in earth dam as isotropic (Fenton and Griffiths 1996) [3], however layered construction may lead to anisotropy. Strong correlation is referred by large scale of fluctuation which means the value of normalized quantity reach to the deterministic value and vise versa. This compatible which theoretical results that as scale of fluctuation reach infinity deterministic value will be identified. (Fenton and Griffiths, [3] 1996, Prospuc et al., 2005) [6].

For very short scales of fluctuation, the flow rate is small, while it increases dramatically as the scale of fluctuation increases Figure (5). At θ_x =0.8 normalized quantity of seepage approximately equals 0.1 which is very low. It shows a good representative flow through dam. At high scale of fluctuation θ_x =10 normalized quantity of seepage approximately equals 0.2 which is greater than that at small scale of fluctuation. The spatial correlation length, also known as the scale of fluctuation, describes the distance over which the spatially random values will tend to be correlated in the underlying Gaussian field. Thus, a large value will imply a smoothly varying field, while a small value will imply a ragged field. For more discussion of the spatial correlation length, the reader is referred to Vanmarcke (1984), Wickremesinghe, (1993) and Jaksa et. al., (1999) [15, 18 , 9].

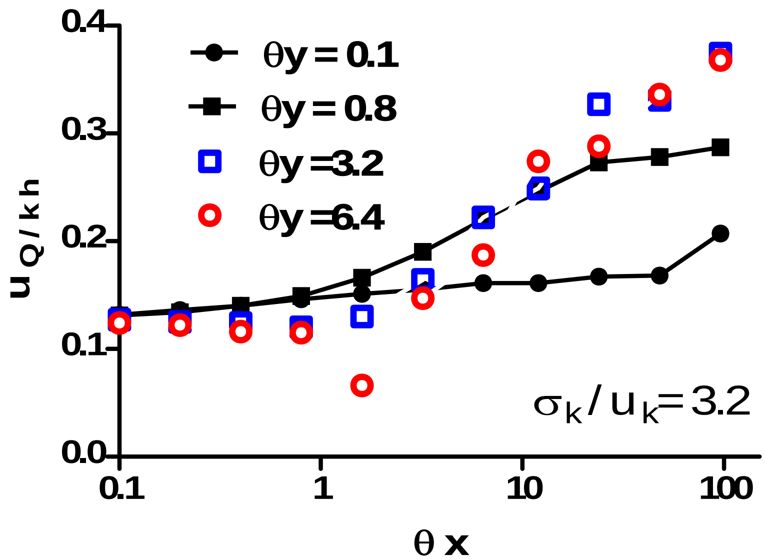

Figure (5). Normalized quantity of seepage versus scale of fluctuation in x direction.

6. Anisotropy of scale of fluctuation

Permeability property in y direction is spatially correlated assuming magnitudes smaller than that in x direction. That can be expressed in term of scale of fluctuation. The scale of fluctuation in vertical direction is smaller than the scale of fluctuation in x direction because of the stratification of soil that makes the water flow as in pipes in the horizontal direction. Figure (6) shows the effect of anisotropy of scale of fluctuation on the result of normalized quantity of flow. The result is somewhat scatter. It is different little from the finding of Ahmed, (2009) [8] who concluded that the quantity of seepage increases as ratio of horizontal to vertical increases. From the study results it was found that at θ_x / θ_y less than unity the normalized quantity increased dramatically and after that the increase in normalized quantity of seepage is slight increase.

Quantity of seepage increases as scale of fluctuation in x direction increases θ_x Figure (6). When horizontal scale of fluctuation is high i.e. strongly correlated (θ_x = 96), quantity of seepage approximately close to deterministic state for case of θ_y = 6.4 i.e. vertical scale of fluctuation slightly increases. For case of small horizontal scale of fluctuation θ_x = 1 for θ_y =0.1, 0.8, 3.2, 6.4, quantity of seepage equal to 1.3. Small horizontal scale of fluctuation means possibility of small permeability for row of cell, that is, make the quantity of seepage small also. The result was presented for σ_k/u_k = 3.2

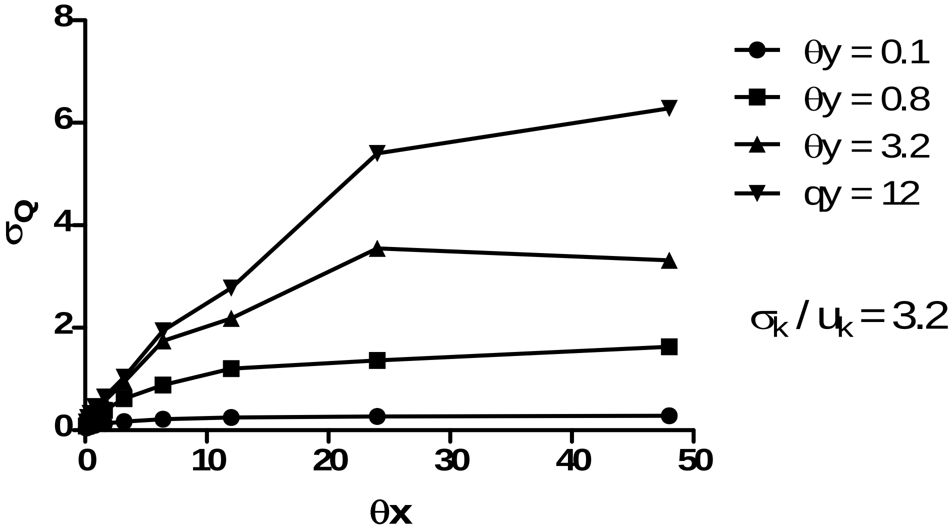

Figure (6). Standard deviation of quantity of seepage versus scale of fluctuation in x direction.

7. Effect of the standard deviation of permeability

Figure (7) shows the relation between flow rate and standard deviation for different scales of fluctuation in x direction. Flow rate changes as standard deviation changes also. The standard deviation of the flow rate increases as flow rate increases until it reaches to value 2.5 then the standard deviation decreases. Scale of fluctuation affects the standard deviation values too much as demonstrated in the graph.

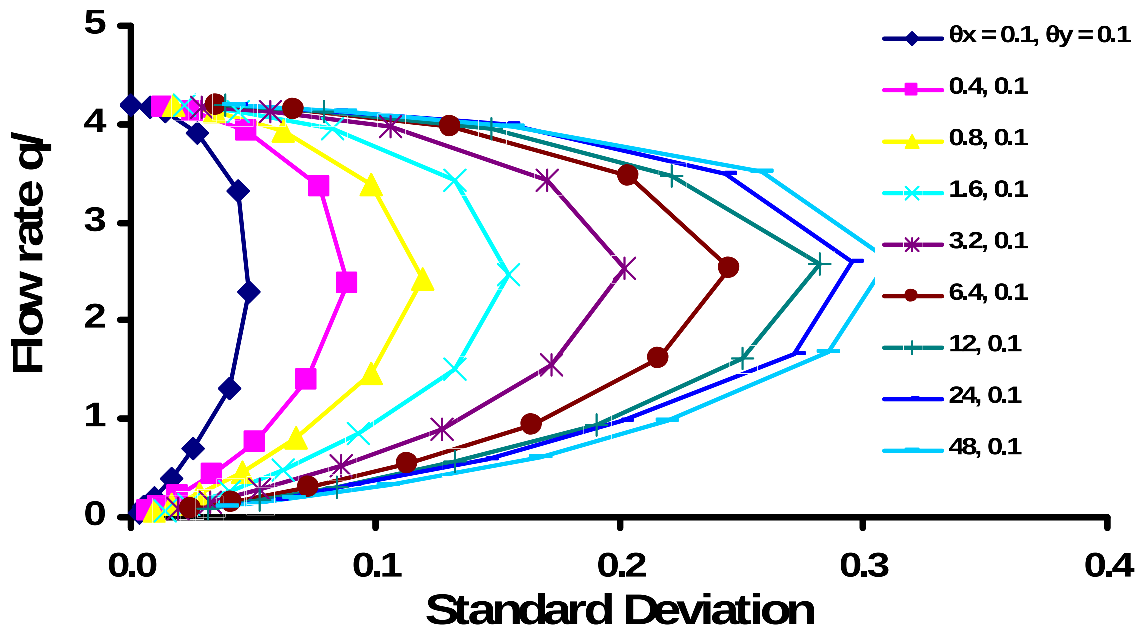

Figure (7). Relation between standard deviation and flow rate.

8. Seepage elevation

Seepage elevation (drawdown) is defined as the elevation of the point on the downstream face of the dam at which the water first reaches the dam surface. This section shows the results of the amount of drawdown of the free surface at the downstream face of the dam. The magnitude of seepage elevation is very important since it determines the height according to which the treatment is desired. The results obtained by executing the program for many times for different value of scale of fluctuation are presented in this section. The relation between seepage elevation and standard deviation is mapped for every percentage of scale of fluctuation Figure (8). Seepage elevation decreases as standard deviation increases. Increasing standard deviation means high difference around the average. High variability may cause reduction in the seepage elevation since existing of small permeability inputs

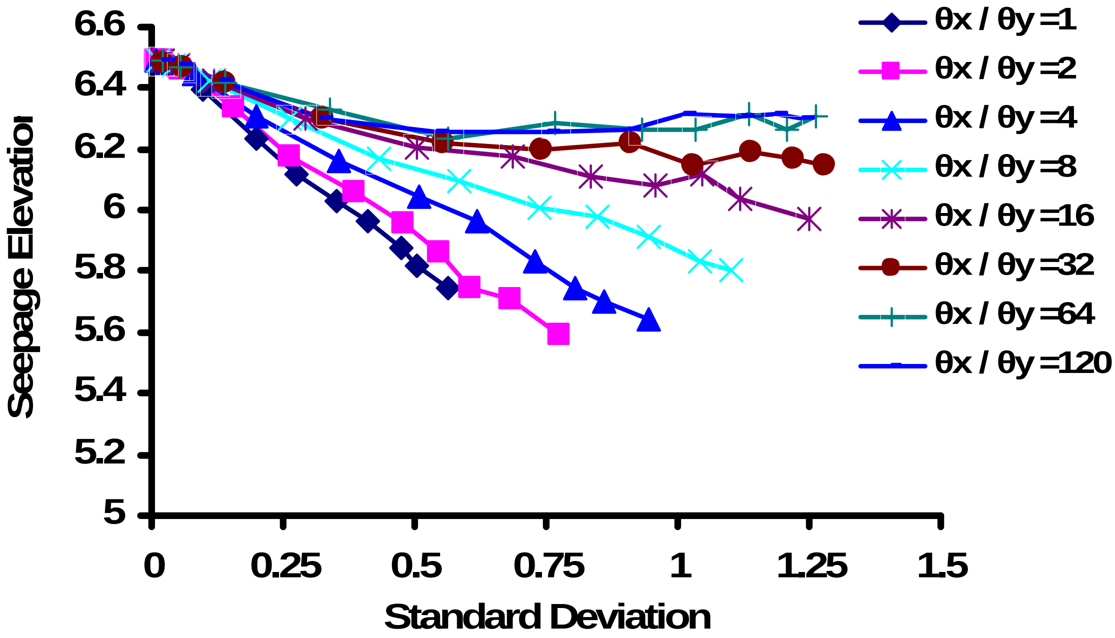

Figure (8). Relation between seepage elevation and standard deviation.

9. Conclusion

Flow through stochastic earth dam has been studied including the effect of anisotropy of scale of fluctuation. The result of research outlines the following conclusions: (1) quantity of seepage through stochastic earth dam is always less than that calculated for deterministic solution which confirms the conclusion of other researchers. This conclusion is very important for design. It means the design on average permeability is in the safe side. (2) It is found that as horizontal to vertical scale of fluctuation (anisotropy of scale of fluctuation increases, quantity of seepage increases but it still less than that for deterministic state (case of average value). (3) Variance of quantity of seepage depends on covariance value. It increases until covariance approximately equals two, then it decreases. (4) Seepage elevation is also decreases as scale of fluctuation decreases.

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11. Nomenclature

Symbols

u	Mean value;
N	Number of events
x	Event
ϕ	Fluid potential
k_x	Permeability in x direction
k_y	Permeability in y direction
σ_{lnk}^2	Variance i.e. second moment of statistics for the normally distributed “ln k”
u_{lnk}^2	Mean i.e. first moment of statistics for the normally distributed “ln k
ρ	Correlation coefficient between two points separated by a large distance τ.
θ_x	Scale of fluctuation in x direction
θ_y	Scale of fluctuation in y direction
σ_k/u_k	Covariance
σ_k	Standard deviation