Dynamic Analysis of non-Prismatic Beam under non-Concentric Axial Force by using The Differential Transformation Method

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Abstract:

In this study, the dynamic analysis of non-prismatic simply-supported Euler-Bernoulli beam subjected to non-concentric compressive axial force has been presented. Kinetic and potential energy expressions of the model are derived. Hamilton's principle is used to obtain the governing equations of motion. The fourth order differential equation with variable coefficient of the beam dynamics and buckling is solved using Differential Transform Method (DTM) to obtain mode shape, natural frequencies and buckling load of the system. The computer package Mathematical is used to write a program to calculate the natural frequencies, buckling load and the mode shapes. The effects of the taper ratio, eccentricity, and bucking load are investigated. The results are compared with the results of the analytical solution where a very good agreement is observed. The results show that the natural frequencies of prismatic and non-prismatic beam decreased when the compressive axial force increased. Also, the results show that when the taper ratio and eccentricity increased, the natural frequency of non-prismatic increases. The buckling load factor decreases when the taper ratio increases.

Keywords: Non-prismatic Euler-Bernoulli Beam, Non-concentric force, Hamilton's Principle, Differential Transform Method

التحليل الديناميكي لعارضة غير منتظمة المقطع معرضة لاحمال ضغط لا محورية باستخدام طريقة النقل التفاضلي

الخلاصة

في هذا البحث تم عرض التحليل الديناميكي لعارضة (Euler-Bernoulli) غير منتظمة المقطع مسندة إسناد بسيط معرضة لأحمال ضغط لا محورية. تم اشتقاق معادلات الطاقات الحركية والكامنة للنموذج. وقد استخدم مبدأ هاملتون للحصول على المعادلات التي تحكم الحركة. معادلة الحركة الديناميكية ومعادلة الانبعاج للعارضة هي معادلة تفاضلية من الدرجة الرابعة وبمعاملات متغيرة تم حلها باستخدام طريقة النقل التفاضلي (DTM) لإيجاد الترددات الطبيعية، أشكال

النسق، حمل الانبعاج للمنظومة. برنامج Mathematical استخدم لكتابة برنامج لحساب كل من الترددات الطبيعية، حمل الانبعاج بالإضافة إلى أشكال النسق. تأثير كل من الانحراف، اللامركزية تم تخمينها. النتائج قورنت مع نتائج التحليل النظري المتوفرة ويلاحظ توافقا جيدا جدا. النتائج بينت بان الترددات الطبيعية تقل بزيادة أحمال الانضغاط للعارضة ذات المقطع المنتظم وغير المنتظم وكذلك بينت النتائج بان الترددات الطبيعية تقل بزيادة نسبة الانحراف و اللامركزية للعارضة غير منتظمة المقطع . معامل حمل الانبعاج يقل بزيادة نسبة الانحراف.

كلمات مرشدة: عارضة (Euler-Bernoulli) غير منتظمة المقطع ، قوة لا محورية ، مبدأ هاملتون ، طريقة النقل التفاضلي.

1. Introduction

Non-prismatic beam are increasingly being used in structures for economic, aesthetic, and other consideration. Design of such structures to resist dynamic forces such as wind and earthquakes, requires knowledge of their natural frequencies and the mode shapes of vibration. The Non-prismatic beam has received great attention from engineers due to their capability in optimizing the strength and weight of the structure.

The vibration problems of non-prismatic beam can be solved by analytic or approximate approaches. The analysis of non-uniform beam vibration using a green function method in the Laplace transformation domain have been investigated by Lee and. Ke [1]. Free vibration of tapered beam with flexible ends using Bessel's' function have been studied by Auciello[2]. Naquleswaran [3] obtained a direct solution for the transverse vibration of Euler-Bernoulli wedge and cone beam. Vibration problems of non-uniform rods and beams using the Rayleigh-Ritz scheme was solved by Abrate.[4]. Rutta P. [5] applied Chebychev series to solve vibration problem for a non-prismatic beam resting on a two parameter non-homogenous elastic foundation. Buckling analysis of non-prismatic columns by using modified vibrational mode shape and energy method was presented by Rahai and Kazemi [6]. Free vibration of non-prismatic beam by the displacement based formulation (stiffness method) was presented by Reza Attarnejad [7].

The complexity in analysis of non-prismatic beam lies in the presence of variable coefficient in the governing differential equation introduced by variable cross-section area and second moment of area. Due to presence of these variables coefficient , exact solutions are generally unavailable except for some special cases, many references mentioned that such as [8,9,10], therefore, a semi-analytical technique based on the Taylor series expansion method which called differential transformation method is using to solve the differential equation.

DTM was applied to solve linear and non-linear initial value problems and partial differential equations by many researches. The concept of DTM was first introduced by Zhou [11] and he used DTM to solve both linear and non-linear initial value problems in electric circuit analysis. Bert and Zeng [12] used DTM to investigate the analysis of axial vibration of compound bars. Numerical solution to buckling analysis of Bernoulli–Euler beams and columns were obtained using DTM and harmonic differential quadrature for various support conditions considering the variation of flexural rigidity by Rajasekaran [13]. The application of techniques of differential transformation method (DTM) to analyze the transverse vibration of a uniform Euler-Bernoulli beam under varying axial force was presented by Young-Jae and Jong-Hak .[14]. The analysis of axially vibrating variable cross-section isotropic rod by using differential transformation method was studied by Mohammed Rafree and Amir Moradi [15]. Transverse vibration of conical Euler-Bernoulli beam using differential transformation method was presented by Torabi et al. [16].

In the present work, the model suggested in Ref. [5] is extended to include the effect of non-concentric compressive axial force on the vibration of non-prismatic beams. The eccentric compressive force can be resolved into a force and a couple moments at the center of the cross section of the beam. The Differential Transform Method (DTM) was used to solve the fourth order differential equation with variable coefficient of the beam vibration and buckling to obtain mode shape, natural frequencies and buckling load of the system. Also, the effect of this force and the effect of eccentricity on fundamental frequency of the non-prismatic beam are conducted.

2. Differential Transform Method

The differential transform method is a semi-analytic transformation technique based on the Taylor series expansion and is a useful tool to obtain analytical solutions of the differential equations. In this method certain transformation rules are applied to both the governing differential equations of motion and the boundary conditions of the system in order to transform them into a set of algebraic equations. The solution of these algebraic equations gives the desired results of the problem. It is different from high-order Taylor series method because Taylor series method requires symbolic computation of the necessary derivatives of the data functions and is expensive for large orders. The basic definitions and the application procedure of this method can be introduced as follows;

Consider a function y(x) which is analytic in a domain D and let x = x_0 represent any point in D Then, the function $y(x)$ can be represented by a power series whose center is located at x_0 and the differential transform of the function $y(x)$ is given by

$$Y[k] = \frac{1}{k!}\left(\frac{d^k y(x)}{dx^k}\right)_{x=x_0} \tag{1}$$

where $y(x)$ is the original function and $Y[k]$ is the transformed function, which is called T-function. The inverse transformation of $Y[k]$is defined as

$$y(x) = \sum_{k=0}^{\infty}(x - x_0)^k Y[k] \tag{2}$$

Combining Eqs. (1) and (2), gives

$$y(x) = \sum_{k=0}^{\infty}\frac{(x - x_0)^k}{k!}\left(\frac{d^k y(x)}{dx^k}\right)_{x=x_0} \tag{3}$$

Considering $f(x)$ by a series of finite terms, Eq. 3 is arranged as follows, with assuming the residual terms to be negligibly small. The increase of convergence is determined by the value q.

$$y(x) = \sum_{k=0}^{q}\frac{(x - x_0)^k}{k!}\left(\frac{d^k y(x)}{dx^k}\right)_{x=x_0} \tag{4}$$

Table 1 shows lists of the transformation properties that are useful in the analysis that follows.

Table 1. DTM theorems used for equations of motion [17]

Original Function	Transformed Function
$f(x) = g(x) \pm h(x)$	$F(k) = G(k) \pm H(k)$
$f(x) = \lambda g(x)$	$F(k) = \lambda G(k)$
$f(x) = g(x)h(x)$	$F(k) = \sum_{l=0}^{k} G(l)H(k-l)$
$f(x) = \frac{d^n g(x)}{dx^n}$	$F(k) = \frac{(k+n)!}{k!}G(k+n)$
$f(x) = x^n$	$F(k) = \delta(k-n) = \begin{cases} 0 & if \quad k \neq n \\ 1 & if \quad k = n \end{cases}$

3. Mathematical model and formulation:

For a non-prismatic Euler-Bernoulli simply supported beam under non-concentric compressive axial force, the governing differential equations of motion is derived by applying Hamilton's principle. Figure (1) shows a schematic representation of the problem under consideration.

The potential or strain energy U of the beam due to bending and axial compressive force is given by;

$$U = \frac{1}{2}\int_0^L EI(x)(y'')^2 dx - \frac{1}{2}N\int_0^L y'^2 dx \tag{5}$$

The kinetic energy T of the beam is given by;

$$T = \frac{1}{2}\int_0^L \rho A(x)\dot{y}^2 dx \tag{6}$$

Hamilton's principle, which is expressed as follows, is applied to the energy expressions given above in order to obtain the governing equations of motion and the boundary conditions

$$\int_{t1}^{t2} \delta(T - U)dt = 0 \tag{7}$$

Where t_1 and t_2 are the time intervals in the dynamic trajectory, and δ is the usual variation operator.

Substituting for T and U from Eqs. (5) and (6) into Eq.(7), using the δ operator, integrating each term by parts, and collecting terms gives the following governing differential equation in free vibration for the non-prismatic beam under non-concentric axial force.

$$\frac{\partial^2}{\partial x^2}(EI(x)\frac{\partial^2 y}{\partial x^2}) + N\frac{\partial^2 y}{\partial x^2} + \rho A(x)\frac{\partial^2 y}{\partial t^2} = 0 \tag{8}$$

The boundary conditions for simply supported beam whose length is L are given as:

$$y(x)=0\ ,\qquad EI(x)\frac{\partial^2 y(x)}{\partial x^2}-M=0 \qquad \text{at}\quad x=0 \tag{9a}$$

$$y(x)=0\ ,\qquad EI(x)\frac{\partial^2 y(x)}{\partial x^2}-M=0 \qquad \text{at}\quad x=l \tag{9b}$$

Where $M = N.e$

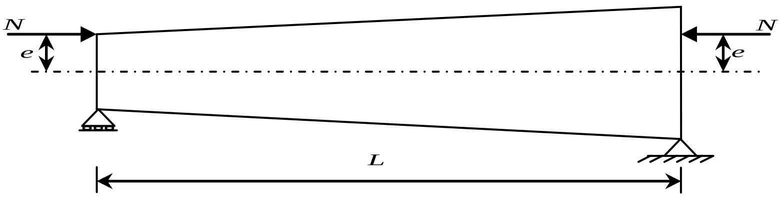

Fig. 1. A schematic diagram

For free vibration analysis of the own problem, let us assume the solution is in the form of sinusoidal variation of $y=Y(x,t)$ with circular frequency ω:

$$y=Y(x)e^{i\omega t} \tag{10}$$

Substituting equation (10) into Eqs. (8), and (9), equation of motion and boundary conditions are expressed as follows:

$$\frac{d^2}{dx^2}(EI(x)\frac{d^2Y(x)}{dx^2})+N\frac{d^2Y(x)}{dx^2}-\rho A(x)\omega^2 Y(x)=0 \qquad x\in(0,l) \tag{11}$$

$$Y(x)=0\ ,\qquad EI(x)\frac{d^2Y(x)}{dx^2}-M=0 \qquad \text{at}\quad x=0 \tag{12a}$$

$$Y(x)=0\ ,\qquad EI(x)\frac{d^2Y(x)}{dx^2}-M=0 \qquad \text{at}\quad x=l \tag{12b}$$

Introduce the following non-dimensional quantities:

$$\xi=\frac{x}{L},\quad \overline{Y}=\frac{Y}{L},\quad \overline{N}=\frac{NL^2}{\pi^2 EI},\quad \mu^2=\omega L^2\sqrt{\frac{\rho A}{EI_\circ}}$$

$$b(\xi)=\frac{I(x)}{I_\circ}\ ,\qquad q(\xi)=\frac{A(x)}{A_o}$$

$$m=\frac{N.e.l}{EI_\circ},\qquad g=\frac{N.e.l}{EI(l)}\ ,\qquad C=\pi^2\overline{N} \tag{13}$$

The governing differential equation and boundary conditions can be rewritten in the following non-dimensional form:

$$\frac{d^2}{d\xi^2}(b(\xi)\frac{d^2\overline{Y}(\xi)}{d\xi^2})+C\frac{d^2\overline{Y}(\xi)}{d\xi^2}-\mu^2 q(\xi)\overline{Y}(\xi)=0 \qquad (14)$$

$$Y(\xi)=0 \quad , \quad \frac{d^2Y(\xi)}{d\xi^2}-m=0 \qquad \text{at } \xi=0 \qquad (15a)$$

$$Y(\xi)=0 \quad , \quad \frac{d^2Y(\xi)}{d\xi^2}-g=0 \qquad \text{at } \xi=1 \qquad (15b)$$

4. Application of DTM

4.1 Free vibration problem of non-prismatic beam

From the definition and properties of DT transformation given in table 1, the DT of the equation of motion (14) after defining $\lambda=\mu^2$ is found as [17],

$$\sum_{r=0}^{k}\overline{B}(k-r)(r+1)(r+2)(r+3)(r+4)\overline{Y}(k+4)+2\sum_{r=0}^{k}(k-r+1)\overline{B}(k-r+1)(r+1)(r+2)(r+3)\overline{Y}(r+3)$$
$$+\sum_{r=0}^{k}(k-r+1)(k-r+2)\overline{B}(k-r+2)(r+1)(r+2)\overline{Y}(r+2)+C(r+1)(r+2)\overline{Y}(r+2) \qquad (16)$$
$$=\sum_{r=0}^{k}\lambda\overline{Q}(k-r)\overline{Y}(r)$$

Where $\overline{B}(k), \overline{Q}(k), and \overline{Y}(k)$ are the T-function of $b(\zeta)$, $q(\zeta)$ and $y(\zeta)$ respectively. Additionally, the differential transform method is applied to Eqs (15a)-(15b) and the following transformed boundary conditions are obtained.

$$\overline{Y}(0)=0, \quad 2\overline{Y}(2)-m=0 \qquad \text{at } \xi=0 \qquad (17a)$$

$$\sum_{k=0}^{m}\overline{Y}(k)=0, \quad \sum_{k=0}^{m}k(k-1)\overline{Y}(k)-g=0 \qquad \text{at } \xi=1 \qquad (17b)$$

The boundary conditions given by Eqs (17a), (17b) and the missing boundary conditions that are assumed to be $\overline{Y}(1)=s$, $\overline{Y}(3)=z$ where s and z are constants, are substituted into Eq. (16). Therefore, the following expression is obtained

$$A_{j1}^{(n)}(\lambda)s + A_{j2}^{(n)}(\lambda)z = 0 \quad , \; j = 1,2,3,..n \tag{18}$$

Where $A_{j1}^{(n)}(\lambda)$, $A_{j2}^{(n)}(\lambda)$ are polynomials of λ corresponding to n.

By Eq. (18), we have the frequency equations as follows.

$$\begin{vmatrix} A_{11}^{n(\lambda)} & A_{12}^{n}(\lambda) \\ A_{21}^{n}(\lambda) & A_{22}^{n}(\lambda) \end{vmatrix} = 0 \tag{19}$$

Solving equation (19), we get $\lambda = \lambda_j^{(n)}$ where $j = 1,2,3,..n$. Here, $\lambda_j^{(n)}$ is the j^{th} estimated eigenvalues corresponding to n. The value of n is obtained by the following equation:

$$\left|\lambda_j^{n} - \lambda_j^{(n-1)}\right| \le \varepsilon \tag{20}$$

where ε is the tolerance parameter. If equation (20) is satisfied, then we have j^{th} eigenvalues $\lambda_j^{(n)}$.

In general, $\lambda_j^{(n)}$ are conjugated complex values, and can be written as $\lambda_j^{(n)} = a_j + ib_j$. Neglecting the small imaginary part b_j, we have the j^{th} natural frequency.

4.2 Buckling problem of non-prismatic beam

When the natural frequency of the system vanishes under the axial loading, the system begins to buckle. By introduction $\mu^2 = 0$ into Eq. (16), one gets the relation

$$\frac{d^2}{d\xi^2}(b(\xi)\frac{d^2\overline{Y}(\xi)}{d\xi^2}) + C\frac{d^2\overline{Y}(\xi)}{d\xi^2} = 0 \tag{21}$$

By applying the DTM to Eq. (22), and using the transformation operation and after some simplifications, the following recurrence equation can be obtained

$$\begin{aligned}&\sum_{r=0}^{k}\overline{B}(k-r)(r+1)(r+2)(r+3)(r+4)\overline{Y}(k+4) + 2\sum_{r=0}^{k}(k-r+1)\overline{B}(k-r+1)(r+1)(r+2)(r+3)\overline{Y}(r+3)\\ &+\sum_{r=0}^{k}(k-r+1)(k-r+2)\overline{B}(k-r+2)(r+1)(r+2)\overline{Y}(r+2) + C(r+1)(r+2)\overline{Y}(r+2) = 0\end{aligned} \tag{22}$$

The differential transformation boundary conditions are the same as that of equations (17a), and (17b) for simply-supported beam.

By assuming $Y(1)=c1$ and $Y(3)=c2$, then Eq. (22) can be calculated up to n terms, and it will

be substituted in Eq. (17), and solving these two equations for non-trivial solutions we get $N_{cr}=N_{cr}^{(n)}$. Here $N_{cr}^{(n)}$ is the estimated buckling load value corresponding to n. The value of n is obtained by the following equation:

$$\left|N_{cr}^{(n)}-N_{cr}^{(n-1)}\right|\leq\varepsilon \tag{23}$$

Where ε is the tolerance parameter and four decimal accuracy considered in the present analysis.

5. Results and discussion

5.1 Free Vibration:

Consider a prismatic beam with modulus of elasticity E = 200 G Pa, beam length L= 3m, rectangular cross-sectional area with 0.1 m height and 0.08 width, subjected to a concentric and non-concentric compressive axial force with different values. The natural frequencies of the beam under concentric force are determined according to Eq. (19), and compared with those of Singiresu S. Rao [18] as tabulated in Table (1). It is seen that the present values show a good agreement with those of Singiresu S. Rao [18]. The dimensionless natural frequency λ corresponding to various dimensionless compressive forces N_r are shown in Fig. (2), where $N_r=Nl^2/\pi^2EI$. As the compressive axial force increased, the natural frequencies of all modes decreased, at a certain compressive force, the critical buckling load, the lowest natural frequency drops to zero and the beams elastically become unstable.

Table 1: natural frequencies of the present study and the exact result Comparison

Axial force (kN.)	ω_1 **Rad/s**	ω_2 **rad/s**	ω_3 **rad/s**	ω_4 **rad/s**
N = 0	160.3046	641.2173	1442.7434	2564.8719
N= 500	130.0443	613.1987	1415.0751	2537.3112
N= 1000	90.1298	583.8353	1386.8454	2509.4612
N= 1000 Ref.(18)	90.1293	583.8313	1386.8320	2509.4392

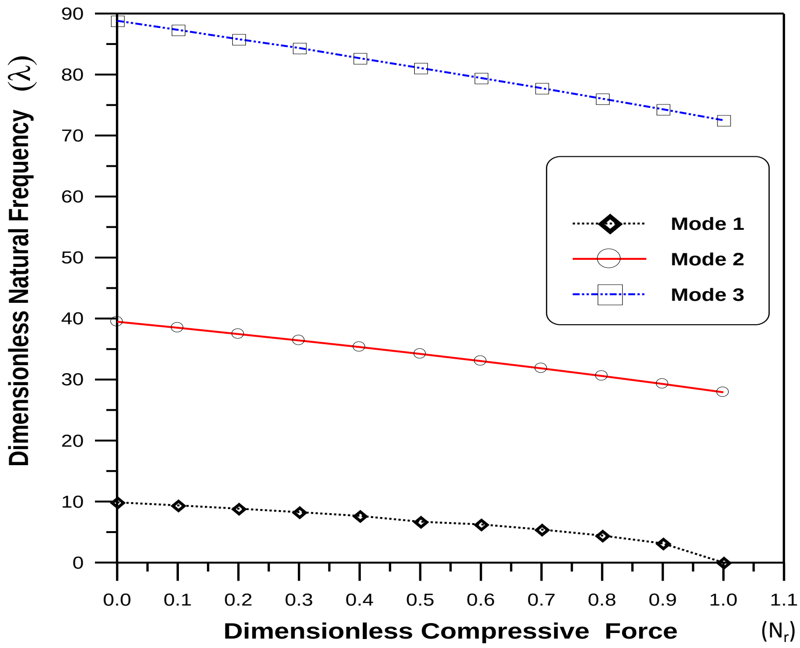

Fig.(2) Dimensionless natural frequency $\omega(\rho A l^4 / EI)^{\frac{1}{2}}$ **versus dimensionless compressive force** $N_r = Nl^2/\pi^2 EI$ **.**

The first three mode shapes for prismatic Euler-Bernoulli simply supported beam without and with compressive axial force are shown in Figs. (3) and (4).

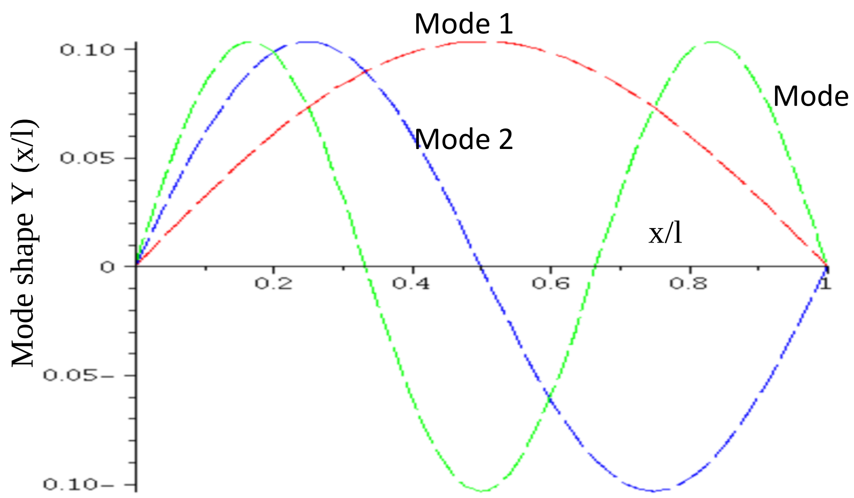

Fig.(3) The first three mode shapes of simply-simply supported beam without compressive axial force

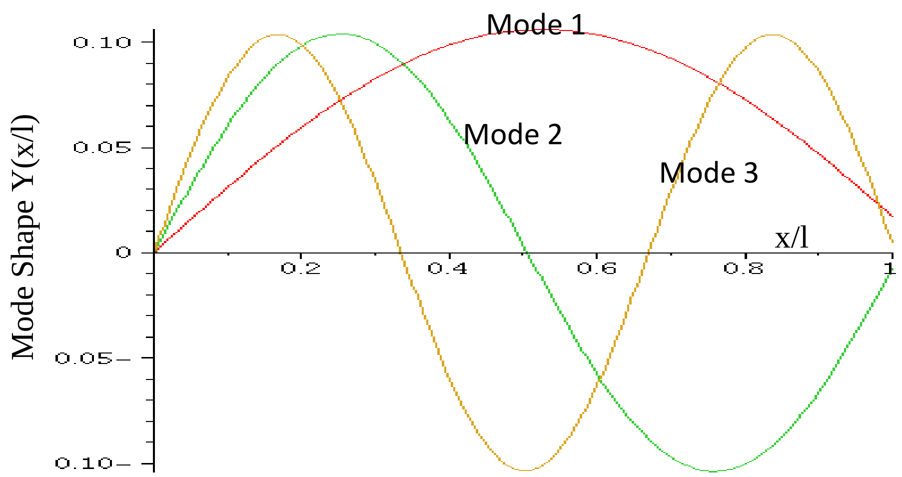

Fig.(4) The first three mode shapes of simply-simply supported beam with compressive axial force (N = 1x10^3 KN)

On the other hand, when the beam is subjected to non-concentric compressive axial force with eccentricity (e), the dimensionless natural frequencies are obtained by using (DTM) for different values of axial compressive force and eccentricities. Figure (5), shows the effect of eccentricity on the natural frequencies of prismatic beam. In this figure, it is seen that the critical concentric axial compressive force is higher compared to that for beam subjected to non- concentric force due to the moment which results from the eccentricity of this force.

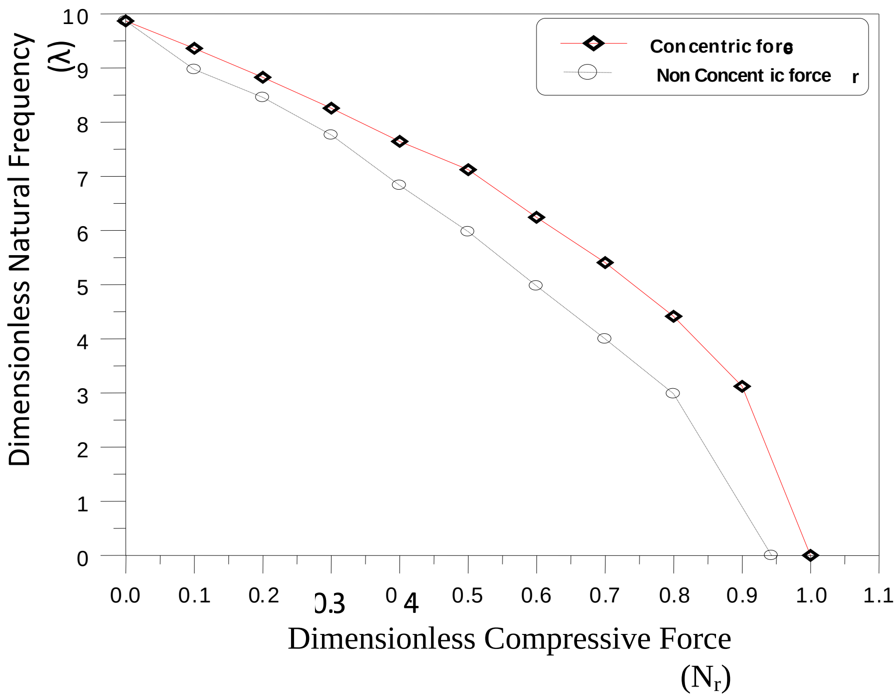

Fig.(5) Variation of dimensionless natural frequency with the dimensionless Compressive axial force

Now, consider a non-prismatic beam with simply-supported boundary conditions is subjected to a concentric and non-concentric compressive axial force. The dimensionless natural frequencies for taper ratio (α = 0, 0.5) for different values of compressive axial force are obtain as tabulated in Table (2). It can be seen that, the natural frequencies decreases when the compressive force increases.

Table (2), Non-dimensional natural frequency of simply supported non-prismatic beam with different non-dimensional axial compressive force

N_r	α=0			α=0.5		
	Ω1	Ω2	Ω3	Ω1	Ω2	Ω3
N_r=0	9.8696	39.4784	88.826	8.081	32.2264	72.7259
N_r=0.25	8.5473	34.1893	76.9259	5.7353	22.9225	51.5734
N_r=0.345	7.9876	31.9507	71.8891	4.5264	18.1045	40.7343
N_r=0.5	6.9788	27.9154	62.8097	0.5983	2.3923	5.3824

5.2 Buckling load

A non-prismatic beam with different second moment of area and different tapered ratio under concentrated compressive force are studied. As a case study, the non-prismatic beam with constant is of and linearly varying height, $I(\xi) = (1-\alpha\xi)^3$. The critical buckling load (N_r) using differential transform methods are calculated and they are listed in Table (3). These results agree very well with these by using finite element method (FEM).

Table (3). Critical buckling load (N_r) for tapered beam with $I(\xi) = (1-\alpha\xi)^3$

α	N_r (DTM)	FEM	Discrepancy %
0	9.870	9.870	0.000
0.2	7.081	7.091	0. 014
0.4	4.685	4.691	0. 012
0.6	2.672	2.674	0. 007
0.8	1.082	1.091	0. 083

Second case study, the non-prismatic beam with constant height and linearly varying width, which means ($I(\xi) = (1-\alpha\xi)$). The results are also evaluated under the simply-supported boundary conditions and compressive axial force. The variation of the critical buckling load with the taper ratio for simply-supported boundary condition and different second moment of area (I) is plotted in Fig.(6) which illustrates the decreasing of the critical load as the taper ratio increased. By comparison of these two curves in Fig.(6), we can see that the critical buckling loads of the second case study are greater than the corresponding values of first for the same taper ratio. This implies that the beam with constant height and linearly varying width has a stronger than that with constant width and linearly varying height. In other words, beam under compressive axial force is easy to buckle towards the linearly varying height direction, rather than linearly varying width direction. Such conclusion is easily understood since the bending stiffness of ($I(\xi) = (1-\alpha\xi)^3$) is less than ($I(\xi) = (1-\alpha\xi)$).

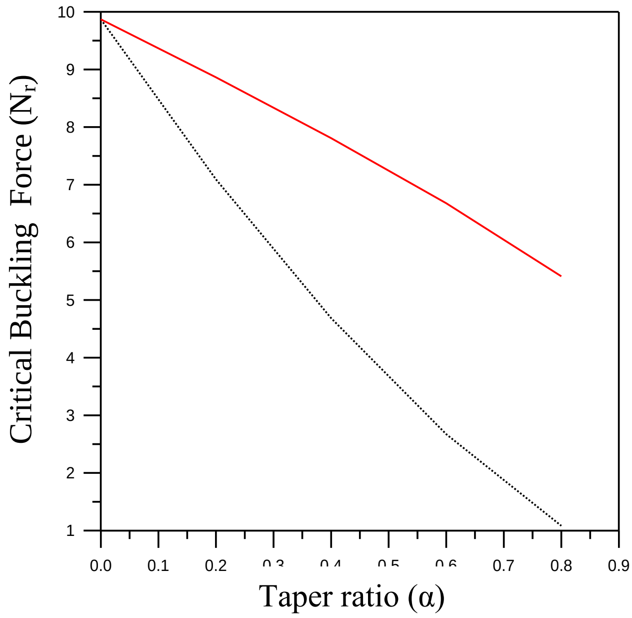

Fig.(6) Variation critical buckling force (N_r) with the taper ratio (α) (—— (I(ξ) = (1-αξ)), (.... I(ξ) = (1-αξ)^3)

6. Conclusions

The main conclusions of the present work can be summarized as:

1- The application of DTM to both the governing equations of motion, buckling, and the boundary conditions are very easy. Moreover, DTM produces simple algebraic equations that can be solved very quickly using the symbolic computational software, Mathematical.

2- The calculated results using differential transformation method (DTM) give good agreement when compared with reference values.

3- The natural frequencies of prismatic and non-prismatic beam decrease when the compressive axial force increases.

4- The result show that when the taper ratio and eccentricity increases, the natural frequency of non-prismatic decreases.

5- The buckling load factor decreases when the taper ratio of non-prismatic beam increases.

6- Beam under compressive axial force is easy to buckle towards the linearly varying height direction, rather than linearly varying width direction.

7. References

[1] S.Y.Lee, H.Y.Ke : Exact solution for the analysis of general elastically restrained non-uniform beam. Transactions of the ASME: Journal of the Applied Mechanics, Vol.59. pp. 205-212, 1992.

[2] M.A.De Rosa, N.M. auciello: Free vibration of tapered beam with flexible ends. Computer and Structure, Vol.60, pp. 197-202, 1996.

[3] S. Naguleswaran. A direct solution for the transverse vibration of Euler-Bernoulli wedge and cone beam. Journal of Sound & Vibration. Vol.172, 289-304 , 1994.

[4] S. Abrate: Vibration of non-uniform rods and beams. Journal of sound and vibration. Vol.187(4), 724-726 ,1995.

[5] P. Ruta: Application of Chebychev series to solution of non-prismatic beam vibration problems. Journal of Sound & Vibration. Vol. 227, 449-467, 1999.

[6] A.R. Rahai, S. Kazemi: Buckling analysis of non-prismatic columns based on modified vibration. Communications in Nonlinear Science and Numerical Simulation, Vol. 13, pp. 1721-1733, 2008.

[7] Attarnejad, R. "Free vibration of non-prismatic beams", Proceedings of 15th ASCE Engineering Mechanics Conference, New York , 2002

[8] Ece, M.C., Aydogdu, M. and Taskin, V. "Vibration of a variable cross-section beam", Mechanics Research Communications, Vol. 34, pp. 78-84. 2007.

[9] R. Attarnejad, A.K. Ghorbani-Tanha, A general fourth order ordinary differential equation with solution in terms of Bessel functions: theory and engineering applications, Struct. Eng. Mech. 27 (2007) 773–776..

[10] M.C. Ece, M. Aydogdu, V. Taskin, Vibration of a variable cross-section beam, Mech. Res. Commun. 34 (2007) 78–84.

[11] Zhou JK. Differential Transformation and its Application for Electrical Circuits. China: Wuhan, Huazhong University Press; 1986.

[12] Bert, C.W., Zeng, H.: Analysis of axial vibration of compound bars by differential transformation method. J. Sound Vibration , **275**, 641–647 (2004)

[13] Rajasekaran, S.: Buckling of fully and partially embedded non-prismatic columns using differential quadrature and differential transformation methods. Struct. Eng. Mech. 28(2), 221–238 (2008).

[14] Young-Jae Shin , Jong-Hak Yun; Transverse vibration of a uniform Euler-Bernoulli beam under varying axial force using differential transformation method, Journal of Mechanical Science and Technology , Vol. 20, No. 2, pp. 191-196, 2006.

[15] Mohammad Rafiee, Amir Moradi ; Analysis of axially vibrating variable cross-section isotropic rod using DTM, International Journal of Applied Engineering Research, Dindigul, Volume 2, No.2, 2011.

[16] K. Torabi, H.Afshari, E. Zafari, Transverse vibration of non-uniform Euler-Bernoulli beam, using differential transformation method (DTM). Applied Mechanical and Materials, Vol. 110-116, pp. 2400-2405, (2012).

[17] Chen CK, Ho SH. Application of differential transformation to eigenvalue problem. J Appl. Mathe. Comput. 1996, 79, 504-510

[18] Singiresu S. Rao " Vibration of Continuous Systems "JOHN WILEY & SONS. INC. 2007.

Nomenclature

A(x)	Cross-sectional area at the position *x*, (m^2)
A_o	Cross-sectional area at x=0, (m^2)
E	Young's modulus (N/m^2)
e	Eccentricity (m)
I(x)	Moment of inertia at distance at x , (m^4)
I_o	Moment of inertia at x=0 , (m^4)
I(l)	Moment of inertia at distance at x=l, (m^4)
L	Beam length, (m)
M	Bending moment (N.m)
N	Non-concentric compressive axial force, (N)
N_r	Nondimensionalized compressive force
U	Potential Energy (Joule)
T	Kinetic Energy (Joule)
t	Time, (second)
$y(x, t)$	Transverse deflection, (m)
x	Longitudinal coordinate
ρ	Mass density of the beam material (kg/m^3)
μ	Non-dimensional natural frequency
α	Taper ratio
ω	Circular natural frequency (rad/s)
i	$\sqrt{-1}$